

Exactly solvable \mathcal{PT} -symmetric Hamiltonian having no Hermitian counterpart

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Abstract

In a recent paper Bender and Mannheim showed that the unequal-frequency fourth-order derivative Pais-Uhlenbeck oscillator model has a realization in which the energy eigenvalues are real and bounded below, the Hilbert-space inner product is positive definite, and time evolution is unitary. Central to that analysis was the recognition that the Hamiltonian H_{PU} of the model is \mathcal{PT} symmetric. This Hamiltonian was mapped to a conventional Dirac-Hermitian Hamiltonian via a similarity transformation whose form was found exactly. The present paper explores the equal-frequency limit of the same model. It is shown that in this limit the similarity transform that was used for the unequal-frequency case becomes singular and that H_{PU} becomes a Jordan-block operator, which is nondiagonalizable and has fewer energy eigenstates than eigenvalues. Such a Hamiltonian has no Hermitian counterpart. Thus, the equal-frequency \mathcal{PT} theory emerges as a distinct realization of quantum mechanics. The quantum mechanics associated with this Jordan-block Hamiltonian can be treated exactly. It is shown that the Hilbert space is complete with a set of nonstationary solutions to the Schrödinger equation replacing the missing stationary ones. These nonstationary states are needed to establish that the Jordan-block Hamiltonian of the equal-frequency Pais-Uhlenbeck model generates unitary time evolution.

I. INTRODUCTION

A decade ago it was discovered that the non-Dirac-Hermitian Hamiltonian $H = p^2 + ix^3$ has an entirely real quantum-mechanical energy spectrum [1]. The reason for this unexpected spectral reality is that this Hamiltonian has an underlying \mathcal{PT} symmetry and that this symmetry is *unbroken*; that is, the energy eigenstates are also eigenstates of the \mathcal{PT} operator. In general, whenever a Hamiltonian has an unbroken \mathcal{PT} symmetry, its energy spectrum is real [2, 3]. Unbroken \mathcal{PT} invariance serves as an alternative to Dirac Hermiticity in quantum theory. Moreover, for any non-Hermitian Hamiltonian that has a complete basis of real-energy eigenstates, there necessarily exists a similarity transformation that brings it to diagonal Hermitian form. Because the transformation is a similarity rather than a unitary one, such unbroken- \mathcal{PT} -symmetric Hamiltonians are really Dirac-Hermitian Hamiltonians written in a disguised form in terms of a skew basis. However, even if a non-Hermitian Hamiltonian has a real eigenspectrum, it is not automatically diagonalizable because it might be of Jordan-block form. In such an event no similarity transformation exists and the \mathcal{PT} sector then emerges as a distinct and self-contained realization of quantum mechanics.

In this paper we study an exactly solvable model, namely, the equal-frequency version of the Pais-Uhlenbeck oscillator [4] in which a distinct \mathcal{PT} realization arises and for which the \mathcal{PT} -symmetric Hamiltonian has no Dirac-Hermitian counterpart. We explore the quantum mechanics associated with such a self-contained \mathcal{PT} quantum realization and give special attention to the lack of completeness of the energy eigenstates that is characteristic of Jordan-block Hamiltonians. Previously, we were motivated to study the unequal-frequency version of the Pais-Uhlenbeck model because of its ghost problem [5]. Here, we study the Jordan-block structure of the equal-frequency model by extending the techniques that were developed in Ref. [5] to resolve the ghost problem in the unequal-frequency version of the model. We show that even though the Pais-Uhlenbeck-model Hamiltonian develops Jordan-block structure in the equal-frequency limit, the unitarity of the theory is not lost.

This paper is organized as follows: Section II provides a brief summary of \mathcal{PT} quantum mechanics and Sec. III reviews the results of Ref. [5] for the unequal-frequency Pais-Uhlenbeck model. In Sec. IV we construct the Fock space associated with the unequal-frequency Pais-Uhlenbeck model. Then, in Sec. V we use the results of Sec. IV to construct the equal-frequency Fock space, and we show that a Jordan-block form for the Hamiltonian

with its incomplete set of energy eigenstates arises in the equal-frequency limit. In Sec. VI we construct the eigenfunctions of the unequal-frequency theory, and in Sec. VII we take their limit to construct the eigenfunctions of the equal-frequency theory. In doing so, we discover what happened to the missing energy eigenstates. In Sec. VIII we present some conclusions and comments. Finally, in the Appendix we discuss the implementation of the Lehmann representation in theories having non-Hermitian Hamiltonians.

II. BRIEF SUMMARY OF \mathcal{PT} QUANTUM MECHANICS

There has been much research during the past few years on \mathcal{PT} quantum mechanics; early references include Refs. [1, 2, 6] and some recent reviews may be found in Refs. [3, 7]. A \mathcal{PT} quantum theory is one whose dynamics is governed by a Hamiltonian H that commutes with the \mathcal{PT} operator. (Here, \mathcal{P} is the parity operator, which performs spatial reflection, and \mathcal{T} is the time-reversal operator.) The appeal of \mathcal{PT} quantum mechanics is that even if a Hamiltonian is not Dirac Hermitian it will still have real energy eigenvalues whenever it is \mathcal{PT} symmetric and all of its eigenstates are also eigenstates of the \mathcal{PT} operator.

For diagonalizable \mathcal{PT} -symmetric Hamiltonians it is convenient to construct an operator \mathcal{C} , which obeys the three simultaneous algebraic operator equations

$$\mathcal{C}^2 = \mathbf{1}, \quad [\mathcal{C}, \mathcal{PT}] = 0, \quad [\mathcal{C}, H] = 0. \quad (1)$$

The first two of these equations are kinematical, while the third is dynamical because it involves the Hamiltonian H . In terms of \mathcal{C} there is a formal construction of an operator $e^{\mathcal{Q}}$:

$$e^{\mathcal{Q}} = \mathcal{C}\mathcal{P}, \quad (2)$$

where the operator \mathcal{Q} is Hermitian in the conventional Dirac sense [3]. Using the operator \mathcal{Q} it is possible (at least in principle) to map the Hamiltonian H to a Dirac-Hermitian Hamiltonian \tilde{H} by means of a similarity transformation of the form [8, 9]:

$$\tilde{H} = e^{-\mathcal{Q}/2} H e^{\mathcal{Q}/2}. \quad (3)$$

While the similarity transformation (3) is isospectral, it is not unitary because \mathcal{Q} is Hermitian rather than anti-Hermitian. Hence, while \tilde{H} and H have the same energy eigenvalues

E_n , their energy eigenkets are not unitarily equivalent. Rather, the eigenkets $|\tilde{n}\rangle$ and $|n\rangle$ of \tilde{H} and H satisfy

$$\tilde{H}|\tilde{n}\rangle = E_n|\tilde{n}\rangle \quad (4)$$

and

$$H|n\rangle = E_n|n\rangle, \quad (5)$$

and are related by the mapping

$$|n\rangle = e^{\mathcal{Q}/2}|\tilde{n}\rangle. \quad (6)$$

The energy eigenbra states corresponding to these eigenkets cannot be obtained from the kets by simple Dirac conjugation. To construct the eigenbra states of H we take the Dirac-Hermitian conjugate of (4):

$$\langle\tilde{n}|\tilde{H} = E_n\langle\tilde{n}|. \quad (7)$$

We then define

$$\langle n| \equiv \langle\tilde{n}|e^{\mathcal{Q}/2} \quad (8)$$

and note that $\langle n|$ is not an eigenbra state of H . Rather, the eigenbra state of H is given by

$$\langle n|e^{-\mathcal{Q}}H = \langle n|e^{-\mathcal{Q}}E_n. \quad (9)$$

The eigenbra and eigenket states of the Dirac-Hermitian Hamiltonian \tilde{H} obey the usual statements of orthogonality, completeness, and Hamiltonian operator reconstruction:

$$\langle\tilde{n}|\tilde{m}\rangle = \delta_{m,n}, \quad (10)$$

$$\sum_n |\tilde{n}\rangle\langle\tilde{n}| = \mathbf{1}, \quad (11)$$

$$\tilde{H} = \sum_n |\tilde{n}\rangle E_n \langle\tilde{n}|. \quad (12)$$

The similarity transformations in (6) and (8) imply that for the non-Hermitian Hamiltonian H the corresponding statements are

$$\langle n|e^{-\mathcal{Q}}|m\rangle = \delta_{m,n}, \quad (13)$$

$$\sum_n |n\rangle\langle n|e^{-\mathcal{Q}} = \mathbf{1}, \quad (14)$$

$$H = \sum_n |n\rangle E_n \langle n|e^{-\mathcal{Q}}. \quad (15)$$

The norm in (13) is relevant for the \mathcal{PT} -symmetric Hamiltonian H , with $\langle n|e^{-\mathcal{Q}}$ rather than $\langle n|$ being the appropriate energy eigenbra. In any \mathcal{PT} theory for which the operator $e^{-\mathcal{Q}}$ exists, there will be a positive norm of the form given in (13) and no states of negative norm (ghost states). Furthermore, because $[H, \mathcal{CPT}] = 0$, the \mathcal{PT} -symmetric Hamiltonian H will generate unitary time evolution even though it is not Hermitian.

To underscore the need for a non-Dirac norm for non-Hermitian Hamiltonians, we recall the connection between the Schrödinger and Heisenberg representations. Specifically, in the Schrödinger representation one introduces time-dependent states and time-independent operators. Ordinarily one does this for a Hermitian Hamiltonian, and it is instructive to see how things change in the non-Hermitian case. We thus consider Schrödinger equations for ket and bra states:

$$i\frac{d}{dt}|\alpha_S(t)\rangle = H|\alpha_S(t)\rangle, \quad -i\frac{d}{dt}\langle\alpha_S(t)| = \langle\alpha_S(t)|H^\dagger. \quad (16)$$

For time-independent H and H^\dagger the solutions to these equations are

$$|\alpha_S(t)\rangle = e^{-iHt}|\alpha_S(0)\rangle, \quad \langle\alpha_S(t)| = \langle\alpha_S(0)|e^{iH^\dagger t}. \quad (17)$$

We introduce a time-independent (Schrödinger) operator A_S with matrix element

$$\langle\alpha_S(t)|A_S|\alpha_S(t)\rangle = \langle\alpha_S(0)|e^{iH^\dagger t}A_S e^{-iHt}|\alpha_S(0)\rangle, \quad (18)$$

and define the time-dependent (Heisenberg) operator

$$A_H(t) = e^{iH^\dagger t}A_S e^{-iHt}. \quad (19)$$

The operator $A_H(t)$ obeys

$$i\frac{d}{dt}A_H(t) = A_H(t)H - H^\dagger A_H(t). \quad (20)$$

Since H and H^\dagger are different when H is not Dirac Hermitian, the time derivative of $A_H(t)$ is not given by the commutator of $A_H(t)$ with H . However, regardless of whether or not the Hamiltonian is Dirac Hermitian, the Hamiltonian *is* the generator of time translations. Thus, in the Heisenberg representation the time derivative of an operator is always given by

$$i\frac{d}{dt}A_H(t) = A_H(t)H - HA_H(t). \quad (21)$$

To construct the Schrödinger states we start from (21) and work backward. Thus, we replace (16) and (17) by

$$i\frac{d}{dt}|\alpha_S(t)\rangle = H|\alpha_S(t)\rangle, \quad -i\frac{d}{dt}\langle\hat{\alpha}_S(t)| = \langle\hat{\alpha}_S(t)|H, \quad (22)$$

so

$$|\alpha_S(t)\rangle = e^{-iHt}|\alpha_S(0)\rangle, \quad \langle\hat{\alpha}_S(t)| = \langle\hat{\alpha}_S(0)|e^{iHt}, \quad (23)$$

where the bra state $\langle\hat{\alpha}_S(t)|$ is not the Dirac conjugate of the ket state $|\alpha_S(t)\rangle$. In (22) we see that H acts to the right on $|\alpha_S(t)\rangle$ and to the left on $\langle\hat{\alpha}_S(t)|$. The appropriate inner product is given by $\langle\hat{\beta}_S(t)|\alpha_S(t)\rangle$. Because of (23) we have

$$\langle\hat{\beta}_S(t)|\alpha_S(t)\rangle = \langle\hat{\beta}_S(0)|e^{iHt}e^{-iHt}|\alpha_S(0)\rangle = \langle\hat{\beta}_S(0)|\alpha_S(0)\rangle, \quad (24)$$

which is the statement of unitary time development. By contrast, the Dirac inner product

$$\langle\beta_S(t)|\alpha_S(t)\rangle = \langle\beta_S(0)|e^{iH^\dagger t}e^{-iHt}|\alpha_S(0)\rangle \neq \langle\beta_S(0)|\alpha_S(0)\rangle \quad (25)$$

is not time independent. Thus, for non-Hermitian Hamiltonians unitarity is achieved by using an inner product that is different from the usual Dirac inner product. For \mathcal{PT} -symmetric Hamiltonians the states $\langle\beta_S(t)|$ and $\langle\hat{\beta}_S(t)|$ are related by $\langle\hat{\beta}_S(t)| = \langle\beta_S(t)|e^{-\mathcal{Q}}$.

Several models have been discussed in the literature for which one can calculate the operator \mathcal{Q} in closed form. Amongst them is the Lee model [10], where by calculating the correct inner product one can show that the model is explicitly ghost free [11]. Examples in which one can find exact expressions for the equivalent Dirac-Hermitian Hamiltonian \tilde{H} in (3) associated with a given \mathcal{PT} -symmetric Hamiltonian may be found in Refs. [12, 13, 14, 15, 16] and in our work in Ref. [5].

We emphasize that if we are given a Dirac-Hermitian Hamiltonian \tilde{H} and we convert it to non-Hermitian form H by means of the similarity transformation $H = e^{\mathcal{Q}/2}\tilde{H}e^{-\mathcal{Q}/2}$, we know that we are dealing with a Hermitian Hamiltonian in disguise. However, if we start with a non-Hermitian Hamiltonian H , we do not immediately know if H is a disguised Dirac-Hermitian Hamiltonian, and the advantage of \mathcal{PT} symmetry is that it provides a diagnostic for determining whether this might in fact be the case. It may happen that the operator \mathcal{Q} simply does not exist, and when this is the case, the \mathcal{PT} -symmetric Hamiltonian will have no Hermitian counterpart. This situation arises when H has some complex eigenvalues (its \mathcal{PT} symmetry is broken), and thus there is obviously no Hermitian counterpart. A

transition from an unbroken to a broken \mathcal{PT} symmetry has actually been observed in recent laboratory optics experiments [17].

However, in this paper we encounter a more serious and fundamental obstacle in trying to construct a Hermitian Hamiltonian \tilde{H} associated with a non-Hermitian \mathcal{PT} -symmetric Hamiltonian H . Specifically, even if the eigenvalues of H are all real, H may be a Jordan-block matrix that has fewer eigenfunctions than eigenvalues, and thus it is not diagonalizable [18]. This is the case with the equal-frequency Pais-Uhlenbeck oscillator model. For this case the operator \mathcal{Q} of the unequal-frequency oscillator model becomes singular in the equal-frequency limit.

III. REVIEW OF THE UNEQUAL-FREQUENCY PAIS-UHLENBECK OSCILLATOR MODEL

The Pais-Uhlenbeck oscillator was introduced in 1950 as a simple model to explore the structure of a quantum system whose Lagrangian depends on acceleration as well as on position and velocity [4]. The Pais-Uhlenbeck action is

$$I_{\text{PU}} = \frac{\gamma}{2} \int dt \left[\ddot{z}^2 - (\omega_1^2 + \omega_2^2) \dot{z}^2 + \omega_1^2 \omega_2^2 z^2 \right], \quad (26)$$

where γ , ω_1 , and ω_2 are all positive constants. Because the action depends on the acceleration, the differential equation of motion

$$\frac{d^4 z}{dt^4} + (\omega_1^2 + \omega_2^2) \frac{d^2 z}{dt^2} + \omega_1^2 \omega_2^2 z = 0 \quad (27)$$

is fourth order.

The Pais-Uhlenbeck oscillator model is interesting because a fourth-order wave equation leads to a Green's function $G(E)$ whose denominator in energy space is quartic in the energy. To find the Green's function, we replace the right side of (27) by a delta-function source term and take the Fourier transform. The result is

$$G(E) = \frac{1}{(E^2 - \omega_1^2)(E^2 - \omega_2^2)}. \quad (28)$$

The advantage of such a Green's function is that it leads to Feynman integrals that are more convergent than the corresponding integrals constructed from propagators having quadratic denominators.

One may worry that there is a price to pay for such good convergence because the form of this Green's function seems to imply the existence of a ghost state when $\omega_1 \neq \omega_2$. The argument goes as follows: In partial fraction form the Green's function

$$G(E) = \frac{1}{\omega_1^2 - \omega_2^2} \left(\frac{1}{E^2 - \omega_1^2} - \frac{1}{E^2 - \omega_2^2} \right) \quad (29)$$

describes the propagation of two kinds of states, one of energy ω_1 and the other of energy ω_2 . Assuming without loss of generality that $\omega_1 > \omega_2$, it appears that ω_2 is associated with a state of *negative* probability because its residue contribution to the propagator is negative. This appears to violate the positivity condition on the weight function of the Lehmann representation. (Recall that when the two-point Green's function is expressed in Lehmann-representation form the requirement that all quantum states have positive Dirac norm implies that the residues of all intermediate propagating states must be strictly positive. See, for example, Ref. [19].) In the past, theories having fourth-order wave equations have been abandoned because they were thought to violate the Lehmann-representation positivity condition.

However, the Pais-Uhlenbeck oscillator was recently revisited [5] and it was shown that there is a realization of the unequal-frequency model in which the Hilbert space actually contains no ghost states. Specifically, in Ref. [5] it was shown that the above Green's-function argument has a subtle flaw, namely the presumption that the Hamiltonian for the model [20, 21, 22]

$$H_{\text{PU}} = \frac{p_x^2}{2\gamma} + p_z x + \frac{\gamma}{2} (\omega_1^2 + \omega_2^2) x^2 - \frac{\gamma}{2} \omega_1^2 \omega_2^2 z^2 \quad (30)$$

is Dirac Hermitian and that its associated norm is the standard Dirac norm. [Because the action of (26) is constrained [20, 21, 22], to construct the Hamiltonian it was necessary to replace \dot{z} by a new and independent variable x in the action (26).] It was shown in Ref. [5] that one should interpret the Hamiltonian H_{PU} as a member of the class of non-Dirac-Hermitian Hamiltonians that are symmetric under combined space reflection \mathcal{P} and time reversal \mathcal{T} . Thus, as explained in Sec. II, it is necessary to replace the Dirac inner product by the inner product that we argue is appropriate for the Pais-Uhlenbeck Hamiltonian.

When we use the norm in (13) for the unequal-frequency Pais-Uhlenbeck model, this model becomes ghost free and unitary. Moreover, because the norm in (13) is not the conventional Dirac norm, the relative negative sign that appears in the Lehmann representation in (29) cannot be interpreted as the residue associated with a negative-norm (ghost) state.

Rather, we show below that this negative sign is associated with an eigenvalue of the \mathcal{C} operator defined in (1). In the Appendix we discuss this point further.

To explore the structure of the unequal-frequency Pais-Uhlenbeck model in detail we make a further partial-fraction decomposition of the $G(E)$ propagator in (29):

$$G(E) = \frac{1}{2\omega_1(\omega_1^2 - \omega_2^2)} \left(\frac{1}{E - \omega_1} - \frac{1}{E + \omega_1} \right) - \frac{1}{2\omega_2(\omega_1^2 - \omega_2^2)} \left(\frac{1}{E - \omega_2} - \frac{1}{E + \omega_2} \right). \quad (31)$$

In (31) there are two pole terms having positive coefficients and two having negative coefficients. Whether or not these negative coefficients are associated with negative residues depends on the way one performs the contour integration in the complex energy plane. For the conventional Feynman contour, where the positive-frequency poles lie below and the negative-frequency poles lie above the real- E axis, contour integration yields the Feynman propagator

$$\begin{aligned} G^F(t) &= -\frac{1}{2\pi i} \int dE e^{-iEt} G(E) \\ &= \frac{\theta(t)}{\omega_1^2 - \omega_2^2} \left(\frac{e^{-i\omega_1 t}}{2\omega_1} - \frac{e^{-i\omega_2 t}}{2\omega_2} \right) + \frac{\theta(-t)}{\omega_1^2 - \omega_2^2} \left(\frac{e^{i\omega_1 t}}{2\omega_1} - \frac{e^{i\omega_2 t}}{2\omega_2} \right). \end{aligned} \quad (32)$$

In (32) $G^F(t)$ describes the forward propagation of two positive-energy particles and the backward propagation of two negative-energy antiparticles and both of the ω_2 -dependent terms have negative coefficients.

To avoid these negative coefficients we can instead choose an unconventional contour for which the poles at ω_1 and $-\omega_2$ are taken to lie below the real E axis and the poles at $-\omega_1$ and ω_2 to lie above it. For this choice, contour integration yields the unconventional propagator

$$\begin{aligned} G^{\text{UNC}}(t) &= -\frac{1}{2\pi i} \int dE e^{-iEt} G(E) \\ &= \frac{\theta(t)}{\omega_1^2 - \omega_2^2} \left(\frac{e^{-i\omega_1 t}}{2\omega_1} + \frac{e^{i\omega_2 t}}{2\omega_2} \right) + \frac{\theta(-t)}{\omega_1^2 - \omega_2^2} \left(\frac{e^{i\omega_1 t}}{2\omega_1} + \frac{e^{-i\omega_2 t}}{2\omega_2} \right). \end{aligned} \quad (33)$$

Now all of the coefficients are positive, but $G^{\text{UNC}}(t)$ describes the forward propagation of one positive-energy particle and one negative-energy antiparticle, and also the backward propagation of one negative-energy antiparticle and one positive-energy particle.

The $G^F(t)$ propagator describes a system whose multiparticle energy spectrum is bounded below [the energy eigenvalues are $E(n_1, n_2) = (n_1 + 1/2)\omega_1 + (n_2 + 1/2)\omega_2$] but some of its poles have negative residues. The poles of $G^{\text{UNC}}(t)$ have positive residues but the energy spectrum

is unbounded below [the energy eigenvalues are $E'(n_1, n_2) = (n_1 + 1/2)\omega_1 - (n_2 + 1/2)\omega_2$] and thus it exhibits forward propagation of negative-energy states. The $G^F(t)$ and $G^{\text{UNC}}(t)$ propagators both seem to have problems, which explains why fourth-order theories are not thought to be viable.

If we evaluate the Feynman path integral to construct the propagator, we obtain $G^F(t)$ directly and do not obtain $G^{\text{UNC}}(t)$. Specifically, for the Pais-Uhlenbeck action (26) the path integral

$$G(z_i, \dot{z}_i, z_f, \dot{z}_f, T) = \int \mathcal{D}(z, \dot{z}) \exp \left\{ i \frac{\gamma}{2} \int_0^T dt \left[\dot{z}^2 - (\omega_1^2 + \omega_2^2) z^2 + \omega_1^2 \omega_2^2 z^2 \right] \right\} \quad (34)$$

taken over all paths having fixed initial and final velocities can be performed analytically [22]. Taking its deep Euclidean time limit, where $e^{-iEt} \rightarrow e^{-E\tau}$, one finds that the low-lying energy eigenvalues are positive: $E = \omega_1$ and $E = \omega_2$. Also, their excitations are just as required in the conventional Feynman contour prescription [22]. One should not expect a Feynman path integral to give an energy spectrum having negative energies because $e^{-E\tau}$ would not be finite at large τ , so we do not consider $G^{\text{UNC}}(t)$ further [23]. Our task then is to find a physically acceptable quantum-mechanical interpretation for $G^F(t)$.

To do this, we consider the Schrödinger eigenvalue problem associated with the Hamiltonian H_{PU} in (30). We set $p_z = -i\partial_z$, $p_x = -i\partial_x$ and obtain

$$\left[-\frac{1}{2\gamma} \frac{\partial^2}{\partial x^2} - ix \frac{\partial}{\partial z} + \frac{\gamma}{2} (\omega_1^2 + \omega_2^2) x^2 - \frac{\gamma}{2} \omega_1^2 \omega_2^2 z^2 \right] \psi_n(z, x) = E_n \psi_n(z, x), \quad (35)$$

which has ground-state energy $E_0 = (\omega_1 + \omega_2)/2$ and corresponding eigenfunction

$$\psi_0(z, x) = \exp \left[\frac{\gamma}{2} (\omega_1 + \omega_2) \omega_1 \omega_2 z^2 + i\gamma \omega_1 \omega_2 zx - \frac{\gamma}{2} (\omega_1 + \omega_2) x^2 \right]. \quad (36)$$

As $z \rightarrow \pm\infty$, this eigenfunction diverges. Thus, in the space of such an eigenfunction the operator $p_z = -i\partial_z$ cannot be Dirac Hermitian. For a canonical commutator of the form $[z, p_z] = i$, one can only associate p_z with a differential operator $-i\partial_z$ when the commutator acts on test functions $\psi(z)$ that are well behaved at large z . Since this is not the case for the eigenfunction $\psi_0(z, x)$, we see that the operator p_z is not Hermitian on the real- z axis.

The eigenfunction $\psi_0(z, x)$ vanishes exponentially rapidly for large $|z|$ when z is imaginary (or, more generally, when z is confined to the two *Stokes wedges* $|\text{Im}(z)| \geq |\text{Re}(z)|$). To exploit this fact, we perform an operator similarity transform of the quantum-mechanical operators z and p_z :

$$y = e^{\pi p_z z/2} z e^{-\pi p_z z/2} = -iz, \quad q = e^{\pi p_z z/2} p_z e^{-\pi p_z z/2} = ip_z. \quad (37)$$

The commutator of the operators y and q still has the canonical form $[y, q] = i$. In terms of the operators y and q the Hamiltonian now takes the form

$$H = \frac{p^2}{2\gamma} - iqx + \frac{\gamma}{2} (\omega_1^2 + \omega_2^2) x^2 + \frac{\gamma}{2} \omega_1^2 \omega_2^2 y^2, \quad (38)$$

where for notational simplicity we have replaced p_x by p .

Since H and H_{PU} are related by a similarity transform, they both describe the same physics, and one can use the Hamiltonian H to explore the structure of the Pais-Uhlenbeck model [24]. In (38) the operators p , x , q , and y are now formally Hermitian on the real- x and real- y axes, but because of the $-iqx$ term H has become complex and is manifestly not Dirac Hermitian. This non-Hermiticity property was not apparent in the original form of the Hamiltonian H_{PU} given in (30), and it is the key to uncovering the structure of the unequal-frequency Pais-Uhlenbeck model.

While H is not Hermitian, with the \mathcal{P} and \mathcal{T} quantum-number assignments

	p	x	q	y
\mathcal{P}	-	-	+	+
\mathcal{T}	-	+	+	-
\mathcal{PT}	+	-	+	-

(39)

we see that H is \mathcal{PT} symmetric. Thus, H can be transformed to a Hermitian Hamiltonian by means of the similarity transformation $\tilde{H} = e^{-\mathcal{Q}/2} H e^{\mathcal{Q}/2}$. In Ref. [5] the operator \mathcal{Q} was calculated exactly:

$$\mathcal{Q} = \alpha pq + \beta xy, \quad \alpha = \frac{1}{\gamma \omega_1 \omega_2} \log \left(\frac{\omega_1 + \omega_2}{\omega_1 - \omega_2} \right), \quad \beta = \alpha \gamma^2 \omega_1^2 \omega_2^2 \quad (40)$$

It leads to

$$\begin{aligned} \tilde{y} &= y \cosh \theta + i(\alpha/\beta)^{1/2} p \sinh \theta, & \tilde{p} &= p \cosh \theta - i(\beta/\alpha)^{1/2} y \sinh \theta, \\ \tilde{x} &= x \cosh \theta + i(\alpha/\beta)^{1/2} q \sinh \theta, & \tilde{q} &= q \cosh \theta - i(\beta/\alpha)^{1/2} x \sinh \theta, \\ \tilde{H} &= \frac{p^2}{2\gamma} + \frac{q^2}{2\gamma \omega_1^2} + \frac{\gamma}{2} \omega_1^2 x^2 + \frac{\gamma}{2} \omega_1^2 \omega_2^2 y^2, \end{aligned} \quad (41)$$

where $\theta = (\alpha\beta)^{1/2}/2$, $\tilde{y} = e^{-\mathcal{Q}/2} y e^{\mathcal{Q}/2}$, etc. The transformed Hamiltonian \tilde{H} is a manifestly positive-definite operator. Equations (13) – (15) now follow directly and we obtain a viable unequal-frequency theory [25].

IV. FOCK SPACE IN THE UNEQUAL-FREQUENCY CASE

To construct the Fock space associated with the unequal-frequency Pais-Uhlenbeck model, we construct the Heisenberg equations of motion associated with H in (38):

$$\dot{y} = -ix, \quad \dot{x} = \frac{p}{\gamma}, \quad \dot{p} = iq - \gamma(\omega_1^2 + \omega_2^2)x, \quad \dot{q} = -\gamma\omega_1^2\omega_2^2y. \quad (42)$$

All four operators y , x , p , and q obey the same fourth-order operator differential equations,

$$\begin{aligned} \frac{d^4y}{dt^4} + (\omega_1^2 + \omega_2^2)\frac{d^2y}{dt^2} + \omega_1^2\omega_2^2y &= 0, & \frac{d^4x}{dt^4} + (\omega_1^2 + \omega_2^2)\frac{d^2x}{dt^2} + \omega_1^2\omega_2^2x &= 0, \\ \frac{d^4p}{dt^4} + (\omega_1^2 + \omega_2^2)\frac{d^2p}{dt^2} + \omega_1^2\omega_2^2p &= 0, & \frac{d^4q}{dt^4} + (\omega_1^2 + \omega_2^2)\frac{d^2q}{dt^2} + \omega_1^2\omega_2^2q &= 0, \end{aligned} \quad (43)$$

which are identical in form to the classical Pais-Uhlenbeck equation of motion.

Despite the presence of the complex number i in H , all the coefficients in the quantum-mechanical equations of motion are real. From (43) each of the dynamical operators can be expanded in terms of a basis consisting of two raising and two lowering operators having $e^{\pm i\omega_1 t}$ and $e^{\pm i\omega_2 t}$ time dependence. From (42) we thus obtain

$$\begin{aligned} y &= -ia_1e^{-i\omega_1 t} + a_2e^{-i\omega_2 t} - i\hat{a}_1e^{i\omega_1 t} + \hat{a}_2e^{i\omega_2 t}, \\ x &= -i\omega_1a_1e^{-i\omega_1 t} + \omega_2a_2e^{-i\omega_2 t} + i\omega_1\hat{a}_1e^{i\omega_1 t} - \omega_2\hat{a}_2e^{i\omega_2 t}, \\ p &= \gamma[-\omega_1^2a_1e^{-i\omega_1 t} - i\omega_2^2a_2e^{-i\omega_2 t} - \omega_1^2\hat{a}_1e^{i\omega_1 t} - i\omega_2^2\hat{a}_2e^{i\omega_2 t}], \\ q &= \gamma\omega_1\omega_2[-\omega_2a_1e^{-i\omega_1 t} - i\omega_1a_2e^{-i\omega_2 t} + \omega_2\hat{a}_1e^{i\omega_1 t} + i\omega_1\hat{a}_2e^{i\omega_2 t}]. \end{aligned} \quad (44)$$

The operators \hat{a}_1 and \hat{a}_2 are *not* the Dirac-Hermitian adjoints of the lowering operators a_1 and a_2 because they are the raising operators for a non-Hermitian Hamiltonian. In terms of the raising and lowering operators, H takes the diagonal form

$$H = 2\gamma(\omega_1^2 - \omega_2^2) (\omega_1^2\hat{a}_1a_1 + \omega_2^2\hat{a}_2a_2) + \frac{1}{2}(\omega_1 + \omega_2), \quad (45)$$

where the operator commutation algebra is given by

$$\begin{aligned} [a_1, \hat{a}_1] &= [2\gamma\omega_1(\omega_1^2 - \omega_2^2)]^{-1}, & [a_2, \hat{a}_2] &= [2\gamma\omega_2(\omega_1^2 - \omega_2^2)]^{-1}, \\ [a_1, a_2] &= 0, & [a_1, \hat{a}_2] &= 0, & [\hat{a}_1, a_2] &= 0, & [\hat{a}_1, \hat{a}_2] &= 0. \end{aligned} \quad (46)$$

In (45) and (46) the relative signs are all positive, so these equations define a standard two-dimensional harmonic oscillator system [26]. The Heisenberg equations of motion for the raising and lowering operators are

$$\dot{a}_1 = i[H, a_1] = -i\omega_1a_1, \quad \dot{\hat{a}}_1 = i\omega_1\hat{a}_1, \quad \dot{a}_2 = -i\omega_2a_2, \quad \dot{\hat{a}}_2 = i\omega_2\hat{a}_2. \quad (47)$$

The solutions to these equations are $a_1(t) = a_1(0)e^{-i\omega_1 t}$, $\hat{a}_1(t) = \hat{a}_1(0)e^{i\omega_1 t}$, $a_2(t) = a_2(0)e^{-i\omega_2 t}$, $\hat{a}_2(t) = \hat{a}_2(0)e^{i\omega_2 t}$, just as is required of any set of raising and lowering operators.

V. FOCK SPACE IN THE EQUAL-FREQUENCY CASE

The apparatus of \mathcal{PT} quantum mechanics, as developed above, is readily implementable for the unequal-frequency Pais-Uhlenbeck model, but the operator \mathcal{Q} , as given in (40), becomes singular in the equal-frequency limit $\omega_1 - \omega_2 \rightarrow 0$. Additionally, in this limit the partial-fraction form of $G(E)$ in (29) is not valid. Thus, the equal-frequency limit of the Pais-Uhlenbeck model must be treated separately from the unequal-frequency case. We will see that because \mathcal{Q} becomes singular, the \mathcal{PT} -symmetric Hamiltonian H develops a nondiagonalizable Jordan-block structure [27]. Because this happens it is no longer possible to construct an equivalent Hermitian Hamiltonian \tilde{H} . Moreover, the \mathcal{PT} sector of the equal-frequency theory becomes an independent and self-contained realization of quantum mechanics. An objective of this paper is to show that in the singular equal-frequency limit the unitarity of the Pais-Uhlenbeck theory is not lost.

To obtain the equal-frequency limit of the Pais-Uhlenbeck theory we must find a basis in which, unlike (45) and (46), the operators and operator algebra are continuous in the limit. To do this we define

$$\omega_1 \equiv \omega + \epsilon, \quad \omega_2 \equiv \omega - \epsilon \quad (48)$$

and introduce the new operators [28]

$$\begin{aligned} a &= a_1 \left(1 + \frac{\epsilon}{2\omega}\right) + ia_2 \left(1 - \frac{\epsilon}{2\omega}\right), & b &= \frac{\epsilon}{2\omega}(a_1 - ia_2), \\ \hat{a} &= \hat{a}_1 \left(1 + \frac{\epsilon}{2\omega}\right) + i\hat{a}_2 \left(1 - \frac{\epsilon}{2\omega}\right), & \hat{b} &= \frac{\epsilon}{2\omega}(\hat{a}_1 - i\hat{a}_2). \end{aligned} \quad (49)$$

These new operators obey the commutation algebra

$$[a, \hat{a}] = \lambda, \quad [a, \hat{b}] = \mu, \quad [b, \hat{a}] = \mu, \quad [b, \hat{b}] = \lambda, \quad [a, b] = 0, \quad [\hat{a}, \hat{b}] = 0, \quad (50)$$

where

$$\lambda = -\frac{\epsilon^2}{16\gamma(\omega^2 - \epsilon^2)\omega^3}, \quad \mu = \frac{2\omega^2 - \epsilon^2}{16\gamma(\omega^2 - \epsilon^2)\omega^3}. \quad (51)$$

In terms of these new operators the unequal-frequency position operator $y(t)$ is

$$y_{\epsilon \neq 0} = e^{-i\omega t} \left[-i(a - b)\cos \epsilon t - \frac{2b\omega}{\epsilon}\sin \epsilon t \right] + e^{i\omega t} \left[-i(\hat{a} - \hat{b})\cos \epsilon t + \frac{2\hat{b}\omega}{\epsilon}\sin \epsilon t \right], \quad (52)$$

while the unequal-frequency Hamiltonian H in (38) is rewritten as

$$H_{\epsilon \neq 0} = 8\gamma\omega^2\epsilon^2(\hat{a}a - \hat{b}b) + 8\gamma\omega^4(2\hat{b}b + \hat{a}b + \hat{b}a) + \omega. \quad (53)$$

The advantage of these new operators is that $H(\epsilon)$ and $y(\epsilon)$, and also $p(\epsilon)$, $x(\epsilon)$, and $q(\epsilon)$, are continuous in the $\epsilon \rightarrow 0$ limit:

$$H_{\epsilon=0} = 8\gamma\omega^4(2\hat{b}b + \hat{a}b + \hat{b}a) + \omega, \quad (54)$$

$$y_{\epsilon=0} = e^{-i\omega t}[-i(a - b) - 2b\omega t] + e^{i\omega t}[-i(\hat{a} - \hat{b}) + 2\hat{b}\omega t]. \quad (55)$$

The commutation relations (50) are also continuous in the limit $\epsilon \rightarrow 0$ and, together with the commutation relations with $H(\epsilon)$, they tend to the gaugelike form

$$\begin{aligned} [a, \hat{a}] &= 0, \quad [a, \hat{b}] = \frac{1}{8\gamma\omega^3}, \quad [b, \hat{a}] = \frac{1}{8\gamma\omega^3}, \quad [b, \hat{b}] = 0, \quad [a, b] = 0, \quad [\hat{a}, \hat{b}] = 0, \\ [H_{\epsilon=0}, \hat{a}] &= \omega(\hat{a} + 2\hat{b}), \quad [H_{\epsilon=0}, a] = -\omega(a + 2b), \\ [H_{\epsilon=0}, \hat{b}] &= \omega\hat{b}, \quad [H_{\epsilon=0}, b] = -\omega b. \end{aligned} \quad (56)$$

Note that when $\epsilon \neq 0$ the states $\hat{a}|\Omega\rangle$ and $\hat{b}|\Omega\rangle$, where $|\Omega\rangle$ is the no-particle vacuum annihilated by a and b , are not eigenstates of $H_{\epsilon \neq 0}$. Rather, the action of the unequal-frequency Hamiltonian on these states is given by

$$\begin{aligned} H_{\epsilon \neq 0}\hat{a}|\Omega\rangle &= \frac{1}{2\omega}[(4\omega^2 + \epsilon^2)\hat{a}|\Omega\rangle + (4\omega^2 - \epsilon^2)\hat{b}|\Omega\rangle], \\ H_{\epsilon \neq 0}\hat{b}|\Omega\rangle &= \frac{1}{2\omega}[\epsilon^2 a^\dagger|\Omega\rangle + (4\omega^2 - \epsilon^2)\hat{b}|\Omega\rangle], \end{aligned} \quad (57)$$

and the Hamiltonian acts in the one-particle sector as the nondiagonal matrix

$$M_{\epsilon \neq 0} = \frac{1}{2\omega} \begin{pmatrix} 4\omega^2 + \epsilon^2 & 4\omega^2 - \epsilon^2 \\ \epsilon^2 & 4\omega^2 - \epsilon^2 \end{pmatrix}. \quad (58)$$

In the one-particle sector the Hamiltonian has two eigenstates

$$H_{\epsilon \neq 0}|2\omega \pm \epsilon\rangle = (2\omega \pm \epsilon)|2\omega \pm \epsilon\rangle, \quad (59)$$

where

$$|2\omega \pm \epsilon\rangle = [\pm\epsilon\hat{a} + (2\omega \mp \epsilon)\hat{b}]|\Omega\rangle. \quad (60)$$

A similarity transformation diagonalizes $M_{\epsilon \neq 0}$:

$$S^{-1} \left(\frac{1}{2\omega} \right) \begin{pmatrix} 4\omega^2 + \epsilon^2 & 4\omega^2 - \epsilon^2 \\ \epsilon^2 & 4\omega^2 - \epsilon^2 \end{pmatrix} S = \begin{pmatrix} 2\omega + \epsilon & 0 \\ 0 & 2\omega - \epsilon \end{pmatrix}, \quad (61)$$

where

$$\begin{aligned} S &= \frac{1}{2\epsilon\omega^{1/2}(2\omega + \epsilon)^{1/2}} \begin{pmatrix} 2\omega + \epsilon & -(4\omega^2 - \epsilon^2)\epsilon \\ \epsilon & (2\omega + \epsilon)\epsilon^2 \end{pmatrix}, \\ S^{-1} &= \frac{1}{2\epsilon\omega^{1/2}(2\omega + \epsilon)^{1/2}} \begin{pmatrix} (2\omega + \epsilon)\epsilon^2 & (4\omega^2 - \epsilon^2)\epsilon \\ -\epsilon & 2\omega + \epsilon \end{pmatrix}. \end{aligned} \quad (62)$$

Now let us examine the limit $\epsilon \rightarrow 0$. From (60) we can see that as $\epsilon \rightarrow 0$ the two eigenstates $|2\omega \pm \epsilon\rangle$ collapse onto one state $\hat{b}|\Omega\rangle$. Thus, $H(\epsilon)$ loses a one-particle eigenstate in this limit. This is to be expected because the commutator $[H_{\epsilon=0}, \hat{a}]$ contains components parallel to both \hat{a} and \hat{b} . To understand why an eigenstate has disappeared, we note that as $\epsilon \rightarrow 0$ the matrix $M(\epsilon)$ in (58) takes the upper-triangular Jordan-block form

$$M_{\epsilon=0} = 2\omega \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (63)$$

The matrix $M_{\epsilon=0}$ possesses two eigenvalues, both equal to 2ω , but it has only one eigenstate because the eigenvector condition

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} c + d \\ d \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} \quad (64)$$

permits only one solution, namely that with $d = 0$. Despite not being Dirac Hermitian and despite having lost an eigenstate, $H_{\epsilon=0}$ continues to be \mathcal{PT} symmetric under the transformations (39), and thus its eigenvalues remain real. Furthermore, as $\epsilon \rightarrow 0$, S and S^{-1} in (62) both become singular, so it is not possible to diagonalize $M_{\epsilon=0}$.

This same pattern occurs for the higher excited states of H . (For example, one can readily show that the three two-particle states collapse onto one common eigenstate as $\epsilon \rightarrow 0$.) To discuss the $\epsilon \rightarrow 0$ limit for arbitrary multiparticle states, we return to the operator \mathcal{Q} in (40) and note that at $t = 0$ we can use (44) to write

$$\begin{aligned} pq + \gamma^2\omega_1^2\omega_2^2xy \\ = i\gamma^2\omega_1\omega_2(\omega_1^2 - \omega_2^2) [(\omega_1 - \omega_2)(a_1a_2 - \hat{a}_1\hat{a}_2) + (\omega_1 + \omega_2)(\hat{a}_1a_2 - \hat{a}_2a_1)]. \end{aligned} \quad (65)$$

In the $\epsilon \rightarrow 0$ limit we thus obtain

$$pq + \gamma^2\omega_1^2\omega_2^2xy \rightarrow 8\gamma^2\omega^5(\hat{b}^2 - b^2 + \hat{b}a - \hat{a}b). \quad (66)$$

The quantity $pq + \gamma^2\omega_1^2\omega_2^2xy$ is well-behaved as $\epsilon \rightarrow 0$, and the singularity acquired by the coefficient α in (40) cannot be canceled. The similarity transform $e^{-\mathcal{Q}/2}$ in (3) also becomes singular.

Since the operator \mathcal{Q} creates two-particle pairs in (65) and (66), $e^{-\mathcal{Q}/2}$ is a singular operator in every multiparticle sector of H , with each such sector developing Jordan-block structure in the limit. The equal-frequency Pais-Uhlenbeck model possesses no equivalent Dirac-Hermitian counterpart; as we show in the next sections, it constitutes a self-contained realization of quantum mechanics that exists in its own right.

VI. EIGENFUNCTIONS OF THE UNEQUAL-FREQUENCY THEORY

The outstanding property of a Hamiltonian in Jordan-block form is that its eigenstates are incomplete. One may ask, where do the other eigenstates go in the equal-frequency limit, and how can one formulate quantum mechanics if the space of energy eigenstates is incomplete? To answer these questions for the Pais-Uhlenbeck model, we must construct the eigenfunctions of the unequal-frequency theory, where there are no Jordan-block structures, and we must then track what happens to the eigenfunctions in the equal-frequency limit.

Noting that the coordinate-space representation of the Hamiltonian (38) is not symmetric because it has a term that acts like $x\partial_y$, we will need to distinguish between a right Hamiltonian, which operates to the right, and a left Hamiltonian, which operates to the left. When acting to the right on well-behaved states, the commutator $[y, q] = i$ is realized by setting $q = -i\partial_y$, but when acting to the left it is necessary to use $q = +i\partial_y$. We thus obtain the right and left Schrödinger equations

$$i\frac{\partial\psi_n^R(x, y, t)}{\partial t} = \left[-\frac{1}{2\gamma}\frac{\partial^2}{\partial x^2} - x\frac{\partial}{\partial y} + \frac{\gamma}{2}(\omega_1^2 + \omega_2^2)x^2 + \frac{\gamma}{2}\omega_1^2\omega_2^2y^2 \right] \psi_n^R(x, y, t), \quad (67)$$

$$-i\frac{\partial\psi_n^L(x, y, t)}{\partial t} = \left[-\frac{1}{2\gamma}\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial y} + \frac{\gamma}{2}(\omega_1^2 + \omega_2^2)x^2 + \frac{\gamma}{2}\omega_1^2\omega_2^2y^2 \right] \psi_n^L(x, y, t). \quad (68)$$

The ground state, whose energy is $E_0 = (\omega_1 + \omega_2)/2$, has right and left eigenfunctions

$$\psi_0^R(x, y, t) = \exp \left[-\frac{\gamma}{2}(\omega_1 + \omega_2)\omega_1\omega_2y^2 - \gamma\omega_1\omega_2yx - \frac{\gamma}{2}(\omega_1 + \omega_2)x^2 - \frac{i}{2}(\omega_1 + \omega_2)t \right], \quad (69)$$

$$\psi_0^L(x, y, t) = \exp \left[-\frac{\gamma}{2}(\omega_1 + \omega_2)\omega_1\omega_2y^2 + \gamma\omega_1\omega_2yx - \frac{\gamma}{2}(\omega_1 + \omega_2)x^2 + \frac{i}{2}(\omega_1 + \omega_2)t \right]. \quad (70)$$

For this ground state we can define a normalization integral of the form

$$\begin{aligned} N_0 &= \int dx dy \psi_0^L(x, y, t) \psi_0^R(x, y, t) \\ &= \int dx dy \exp \left[-\gamma(\omega_1 + \omega_2)\omega_1\omega_2y^2 - \gamma(\omega_1 + \omega_2)x^2 \right] = \frac{\pi}{\gamma(\omega_1 + \omega_2)(\omega_1\omega_2)^{1/2}}, \end{aligned} \quad (71)$$

with N_0 being time independent, finite, and real.

The two one-particle states with energies $E_1 = E_0 + \omega_1$, $E_2 = E_0 + \omega_2$ have eigenfunctions

$$\begin{aligned}\psi_1^R(x, y, t) &= (x + \omega_2 y) \psi_0^R(x, y, t) e^{-i\omega_1 t}, & \psi_1^L(x, y, t) &= (x - \omega_2 y) \psi_0^L(x, y, t) e^{i\omega_1 t}, \\ \psi_2^R(x, y, t) &= (x + \omega_1 y) \psi_0^R(x, y, t) e^{-i\omega_2 t}, & \psi_2^L(x, y, t) &= (x - \omega_1 y) \psi_0^L(x, y, t) e^{i\omega_2 t},\end{aligned}\quad (72)$$

whose normalization integrals are

$$N_1 = \frac{\pi(\omega_1 - \omega_2)}{2\gamma^2(\omega_1 + \omega_2)^2 \omega_1^{3/2} \omega_2^{1/2}}, \quad N_2 = -\frac{\pi(\omega_1 - \omega_2)}{2\gamma^2(\omega_1 + \omega_2)^2 \omega_1^{1/2} \omega_2^{3/2}}. \quad (73)$$

Again, these normalizations are time independent, finite, and real, but N_2 is negative, a crucial issue that we return to and resolve below.

Analogously, the three two-particle states with energies $E_3 = E_0 + 2\omega_1$, $E_4 = E_0 + \omega_1 + \omega_2$, and $E_5 = E_0 + 2\omega_2$ have right and left eigenfunctions

$$\begin{aligned}\psi_3^R(x, y, t) &= \left[(x + \omega_2 y)^2 - \frac{1}{2\gamma\omega_1} \right] \psi_0^R(x, y, t) e^{-2i\omega_1 t}, \\ \psi_4^R(x, y, t) &= \left[(x + \omega_1 y)(x + \omega_2 y) - \frac{1}{\gamma(\omega_1 + \omega_2)} \right] \psi_0^R(x, y, t) e^{-i\omega_1 t - i\omega_2 t}, \\ \psi_5^R(x, y, t) &= \left[(x + \omega_1 y)^2 - \frac{1}{2\gamma\omega_2} \right] \psi_0^R(x, y, t) e^{-2i\omega_2 t},\end{aligned}\quad (74)$$

and

$$\begin{aligned}\psi_3^L(x, y, t) &= \left[(x - \omega_2 y)^2 - \frac{1}{2\gamma\omega_1} \right] \psi_0^L(x, y, t) e^{2i\omega_1 t}, \\ \psi_4^L(x, y, t) &= \left[(x - \omega_1 y)(x - \omega_2 y) - \frac{1}{\gamma(\omega_1 + \omega_2)} \right] \psi_0^L(x, y, t) e^{i\omega_1 t + i\omega_2 t}, \\ \psi_5^L(x, y, t) &= \left[(x - \omega_1 y)^2 - \frac{1}{2\gamma\omega_2} \right] \psi_0^L(x, y, t) e^{2i\omega_2 t}.\end{aligned}\quad (75)$$

The normalization integrals are time independent:

$$\begin{aligned}N_3 &= \frac{\pi(\omega_1 - \omega_2)^2}{2\gamma^3(\omega_1 + \omega_2)^3(\omega_1\omega_2)^{1/2}\omega_1^2}, \\ N_4 &= -\frac{\pi(\omega_1 - \omega_2)^2}{4\gamma^3(\omega_1 + \omega_2)^3(\omega_1\omega_2)^{1/2}\omega_1\omega_2}, \\ N_5 &= \frac{\pi(\omega_1 - \omega_2)^2}{2\gamma^3(\omega_1 + \omega_2)^3(\omega_1\omega_2)^{1/2}\omega_2^2}.\end{aligned}\quad (76)$$

Note that N_4 is negative.

To relate these normalization integrals to the Hilbert-space norms of the eigenstates of H , we recall that the energy eigenstates of the Hamiltonian are eigenstates of both the

\mathcal{PT} operator and of the $\mathcal{C} = e^{\mathcal{Q}}\mathcal{P}$ operator introduced in Sec. II. The general procedure for constructing such states for symmetric Hamiltonians is given in Ref. [3], and we adapt it here for the nonsymmetric case. Since the \mathcal{PT} operator is antilinear, in general its eigenstates have eigenvalue $e^{i\alpha}$, where α is a real phase that depends on the eigenstate. Multiplying each eigenfunction by $e^{i\alpha/2}$ gives a new eigenfunction that is still an eigenstate of the Hamiltonian, but now it has \mathcal{PT} eigenvalue equal to one. Because the operator \mathcal{C} obeys the algebraic conditions in (1), its eigenstates are also eigenstates of \mathcal{PT} and of H and all of its eigenvalues \mathcal{C}_n are ± 1 . Consequently, each energy eigenstate is an eigenstate of \mathcal{CPT} with eigenvalue $\mathcal{C}_n = \pm 1$.

When the Hamiltonian H is symmetric, the completeness and normalization conditions have the form [3]

$$\begin{aligned} \sum_n e^{iE_n t} [\mathcal{CPT} \psi_n(x', y', t=0)] \psi_n(x, y, t=0) e^{-iE_n t} &= \delta(x - x') \delta(y - y'), \\ \int dx dy e^{iE_n t} [\mathcal{CPT} \psi_n(x, y, t=0)] \psi_m(x, y, t=0) e^{-iE_n t} &= \delta_{m,n}, \end{aligned} \quad (77)$$

where the summation is taken over all the eigenstates of the Hamiltonian. For the Pais-Uhlenbeck oscillator, H is not symmetric and thus we must distinguish between right and left wave functions. We identify $\psi_n^L(x', y', t=0) = \mathcal{CPT} \psi_n^R(x', y', t=0)$ and then assign the \mathcal{C}_n eigenvalues as follows: We take $\mathcal{C}_0 = 1$ for the ground state $\psi_0(x, y, t)$, $\mathcal{C}_1 = 1$ for the one-particle state $\psi_1(x, y, t)$ and $\mathcal{C}_2 = -1$ for the one-particle state $\psi_2(x, y, t)$. The three two-particle states in (74) then acquire eigenvalues $\mathcal{C}_3 = 1$, $\mathcal{C}_4 = -1$, $\mathcal{C}_5 = 1$.

The alternations in sign of the \mathcal{C}_n parallel the alternations in sign of the normalization integrals given in (71), (73), and (76), with the same pattern repeating for the higher multiparticle states; that is, negative signs occur when the number of ω_2 quanta is odd. Thus, for any eigenstate the sign of \mathcal{C}_n is precisely the same as that of its normalization integral $N_n = \int dx dy \psi_n^L(x, y, t) \psi_n^R(x, y, t)$. Consequently, from (77) the correct completeness relation and normalization conditions for the states of the theory are

$$\sum_n \psi_n^L(x', y', t) \frac{\mathcal{C}_n}{|N_n|} \psi_n^R(x, y, t) = \delta(x - x') \delta(y - y'), \quad (78)$$

$$\int dx dy \psi_n^L(x, y, t) \frac{\mathcal{C}_n}{|N_n|} \psi_m^R(x, y, t) = \delta_{m,n}. \quad (79)$$

Both this \mathcal{CPT} norm and the Fock space norm $\langle n | e^{-\mathcal{Q}} | m \rangle = \delta_{m,n}$ in (13) are positive. Thus, there are no negative-norm states in the unequal-frequency Fock space and the unequal-frequency Pais-Uhlenbeck model is unitary.

Because the unequal-frequency energy eigenstates are complete, we can expand an arbitrary wave function in terms of them as $\psi^R(x, y, t) = \sum_n a_n \psi_n^R(x, y, t)$ and $\psi^L(x, y, t) = \sum_n a_n^* \psi_n^L(x, y, t)$. Then, since the energy eigenstates form an orthonormal basis with real energy eigenvalues, it follows that the norm $\int dx dy \psi^L(x, y, t) \mathcal{C} \psi^R(x, y, t)$ is both time independent and real. Therefore, probability is preserved and the unequal-frequency theory is unitary under time evolution.

Given the norm in (79) and writing the norm in (13) in the form $\langle n | e^{-\mathcal{Q}} | m \rangle = \int dx dy \langle n | e^{-\mathcal{Q}} | x, y \rangle \langle x, y | m \rangle$, we make the identifications

$$\frac{\mathcal{C}_n}{|N_n|^{1/2}} \psi_n^L(x, y, t) = \langle n | e^{-\mathcal{Q}} | x, y, t \rangle, \quad \frac{1}{|N_n|^{1/2}} \psi_n^R(x, y, t) = \langle x, y, t | n \rangle. \quad (80)$$

Then, by using the relations in (80) and substituting $H = \sum_n |n\rangle E_n \langle n| e^{-\mathcal{Q}}$ into the expression $G(x, y, x', y', t) = \langle x, y, t = 0 | e^{-iHt} | x', y', t = 0 \rangle$ for the propagator, we obtain

$$G(x, y, x', y', t) = \sum_n \psi_n^R(x, y, t = 0) \frac{\mathcal{C}_n}{|N_n|} e^{-iE_n t} \psi_n^L(x', y', t = 0). \quad (81)$$

Thus, the negative sign of \mathcal{C}_2 accounts for the negative sign of the ω_2 -dependent term in the propagators of (29) and (32). This shows that the good convergence associated with fourth-order propagators can be achieved in a Hilbert space whose norms are all positive and need not be in conflict with the requirement of unitarity.

VII. EIGENFUNCTIONS OF THE EQUAL-FREQUENCY THEORY

In the equal-frequency limit the Hamiltonian (38) takes the form

$$H = \frac{p^2}{2\gamma} - iqx + \gamma\omega^2 x^2 + \frac{\gamma}{2}\omega^4 y^2. \quad (82)$$

Its ground state has energy $E_0 = \omega$, left and right eigenfunctions

$$\begin{aligned} \hat{\psi}_0^R(x, y, t) &= \exp \left[-\gamma\omega^3 y^2 - \gamma\omega^2 yx - \gamma\omega x^2 - i\omega t \right], \\ \hat{\psi}_0^L(x, y, t) &= \exp \left[-\gamma\omega^3 y^2 + \gamma\omega^2 yx - \gamma\omega x^2 + i\omega t \right], \end{aligned} \quad (83)$$

and a normalization integral

$$\hat{N}_0 = \int dx dy \hat{\psi}_0^L(x, y, t) \hat{\psi}_0^R(x, y, t) = \frac{\pi}{2\gamma\omega^2}, \quad (84)$$

which is time independent, finite, and real.

The equal-frequency theory differs from the unequal-frequency theory in that there is only a single one-particle eigenstate instead of two one-particle eigenstates. The energy of this state is $E_1 = 2\omega$ and its eigenfunction is

$$\hat{\psi}_1^R(x, y, t) = (x + \omega y)\hat{\psi}_0^R(x, y, t)e^{-i\omega t}, \quad \hat{\psi}_1^L(x, y, t) = (x - \omega y)\hat{\psi}_0^L(x, y, t)e^{i\omega t}. \quad (85)$$

In the equal-frequency limit the two unequal-frequency eigenstates $\psi_1^R(x, y, t)$ and $\psi_2^R(x, y, t)$ of (72) collapse onto one state $\hat{\psi}_1^R(x, y, t) = [\psi_1^R(x, y, t) + \psi_2^R(x, y, t)]/2$ (and likewise for the left eigenfunction).

The disappearance of eigenstates in the equal-frequency limit is generic. This same collapse of eigenstates occurs for the higher excited eigenstates, with the three unequal-frequency two-particle eigenfunctions in (74) and (75) collapsing onto a single equal-frequency second excited eigenstate having energy $E_2 = 3\omega$:

$$\begin{aligned} \hat{\psi}_2^R(x, y, t) &= \left[(x + \omega y)^2 - \frac{1}{2\gamma\omega} \right] \hat{\psi}_0^R(x, y, t)e^{-2i\omega t}, \\ \hat{\psi}_2^L(x, y, t) &= \left[(x - \omega y)^2 - \frac{1}{2\gamma\omega} \right] \hat{\psi}_0^L(x, y, t)e^{2i\omega t}. \end{aligned} \quad (86)$$

Even though the normalization integral \hat{N}_0 for the equal-frequency ground state is positive, for the equal-frequency first excited state we find that

$$\hat{N}_1 = \int dx dy \hat{\psi}_1^L(x, y, t)\hat{\psi}_1^R(x, y, t) = \int dx dy (x^2 - \omega^2 y^2)e^{-2\gamma\omega^3 y^2 - 2\gamma\omega x^2} = 0. \quad (87)$$

The norm of this state vanishes because the unequal-frequency normalization integrals N_1 and N_2 in (73) both vanish in the equal-frequency limit, and the unequal-frequency states of (72) are orthogonal before the limit is taken. Thus, $\hat{\psi}_1^R(x, y, t)$ ends up being orthogonal to itself, that is, to $\hat{\psi}_1^L(x, y, t)$. This same situation repeats for the higher excited states with all of the two-particle norms of (76) collapsing onto $\hat{N}_2 = 0$.

The emergence of zero-norm states is characteristic of Hamiltonians having Jordan-block structure. For the two-dimensional Jordan-block matrix (64), the right eigenvector is given by the column (1, 0), while the left eigenvector is given by the row (0, 1) rather than by the row (1, 0). Thus, the norm, which is the product of the left and right eigenvectors, is zero. Yet, despite the presence of zero norms, we will show below that probabilities in the equal-frequency theory are still nonzero.

The energy eigenstates that disappear in the equal-frequency limit are replaced by an equal number of nonstationary states. To demonstrate this phenomenon, we form linear

combinations of the unequal-frequency eigenfunctions $\psi_1^R(x, y, t)$ and $\psi_2^R(x, y, t)$ with coefficients that behave like $1/(\omega_1 - \omega_2)$ and then track the limit. To do this, we extract the terms in $\psi_1^R(x, y, t)$ and $\psi_2^R(x, y, t)$ that are linear in $\omega_1 - \omega_2$. This yields the nonstationary equal-frequency state

$$\begin{aligned}\hat{\psi}_{1a}^R(x, y, t) &= \lim_{\epsilon \rightarrow 0} \frac{\psi_2^R(x, y, t) - \psi_1^R(x, y, t)}{2\epsilon} = [(x + \omega y)it + y] \hat{\psi}_0^R(x, y, t) e^{-i\omega t}, \\ \hat{\psi}_{1a}^L(x, y, t) &= \lim_{\epsilon \rightarrow 0} \frac{\psi_2^L(x, y, t) - \psi_1^L(x, y, t)}{2\epsilon} = [-(x - \omega y)it - y] \hat{\psi}_0^L(x, y, t) e^{i\omega t}.\end{aligned}\quad (88)$$

Since this state is not stationary, it is not an eigenstate of H . However, it does satisfy the *time-dependent* equal-frequency Schrödinger equation:

$$\begin{aligned}i\frac{\partial}{\partial t}\hat{\psi}^R(x, y, t) &= \left(-\frac{1}{2\gamma}\frac{\partial^2}{\partial x^2} - x\frac{\partial}{\partial y} + \gamma\omega^2x^2 + \frac{\gamma}{2}\omega^4y^2\right)\hat{\psi}^R(x, y, t), \\ -i\frac{\partial}{\partial t}\hat{\psi}^L(x, y, t) &= \left(-\frac{1}{2\gamma}\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial y} + \gamma\omega^2x^2 + \frac{\gamma}{2}\omega^4y^2\right)\hat{\psi}^L(x, y, t).\end{aligned}\quad (89)$$

Similarly, on expanding the unequal-frequency two-particle states of (74) to order $(\omega_1 - \omega_2)^2$, one obtains [in addition to $\hat{\psi}_2^R(x, y, t)$] the two states

$$\begin{aligned}\hat{\psi}_{2a}^R(x, y, t) &= \lim_{\epsilon \rightarrow 0} \frac{\psi_5^R(x, y, t) - \psi_3^R(x, y, t)}{2\epsilon} \\ &= \left[\left((x + \omega y)^2 - \frac{1}{2\gamma\omega}\right)2it + 2xy + 2\omega y^2 - \frac{1}{2\gamma\omega^2}\right]\hat{\psi}_0^R(x, y, t)e^{-2i\omega t},\end{aligned}\quad (90)$$

and

$$\begin{aligned}\hat{\psi}_{2b}^R(x, y, t) &= \lim_{\epsilon \rightarrow 0} \frac{2\psi_4^R(x, y, t) - \psi_3^R(x, y, t) - \psi_5^R(x, y, t)}{2\epsilon^2} \\ &= \left[\left((x + \omega y)^2 - \frac{1}{2\gamma\omega}\right)2t^2 - \left(2xy + 2\omega y^2 - \frac{1}{2\gamma\omega^2}\right)2it - 2y^2 + \frac{1}{2\gamma\omega^3}\right] \\ &\quad \times \hat{\psi}_0^R(x, y, t)e^{-2i\omega t}.\end{aligned}\quad (91)$$

Both of these states satisfy the equal-frequency time-dependent Schrödinger equation even though neither state is stationary.

The picture is now clear. The unequal-frequency states $\psi_1^R(x, y, t)$ and $\psi_2^R(x, y, t)$ are energy eigenstates, but linear combinations of them are not because the states are not degenerate. However, such linear combinations are still solutions to the unequal-frequency time-dependent Schrödinger equation. When $\epsilon \rightarrow 0$, we obtain two new wave functions $\hat{\psi}_1^R(x, y, t)$ and $\hat{\psi}_{1a}^R(x, y, t)$, which are solutions to the equal-frequency time-dependent Schrödinger equation. One state is stationary, so it also solves the equal-frequency time-independent

Schrödinger equation and thus it is an energy eigenstate, while the other state is not. The same pattern repeats for the higher excited states. The equal-frequency Hamiltonian is a Jordan-block matrix that has fewer eigenstates than eigenvalues, and the form of the unequal-frequency Hamiltonian becomes Jordan block as $\epsilon \rightarrow 0$. The counting of wave functions is continuous in the limit, and it is just the counting of stationary states that is not continuous. The nonstationary states replace the missing stationary states in the limit [29].

Instead of counting energy eigenstates, if we count Fock states, we see that the unequal-frequency Fock space built from the raising operators \hat{a}_1 and \hat{a}_2 in (46) and the equal-frequency Fock space built from the raising operators \hat{a} and \hat{b} in (56) both have the dimensionality of a two-dimensional harmonic oscillator. The eigenspectrum of the equal-frequency Hamiltonian $H_{\epsilon=0}$ is like that of a one-dimensional harmonic oscillator. However, the counting of states in the full Fock space is continuous in the limit $\epsilon \rightarrow 0$.

Given the form of the Schrödinger equation in (89), any pair of its solutions $\hat{\psi}_A^R(x, y, t)$ and $\hat{\psi}_B^L(x, y, t)$ obey

$$i \frac{\partial}{\partial t} \int dx dy \hat{\psi}_B^L(x, y, t) \hat{\psi}_A^R(x, y, t) = - \int dx dy x \frac{\partial}{\partial y} [\hat{\psi}_B^L(x, y, t) \hat{\psi}_A^R(x, y, t)]. \quad (92)$$

Because the product $\hat{\psi}_0^L(x, y, t) \hat{\psi}_0^R(x, y, t) = \exp(-2\gamma\omega^3 y^2 - 2\gamma\omega x^2)$ is exponentially suppressed, the overlap integrals for any pair of multiparticle eigenfunctions are time independent:

$$i \frac{\partial}{\partial t} \int dx dy \hat{\psi}_B^L(x, y, t) \hat{\psi}_A^R(x, y, t) = 0, \quad (93)$$

where the indices A and B are 0, 1, 1a, 2, 2a, 2b, and so on. (These eigenfunctions have the form of polynomials in x and y multiplied by the ground-state eigenfunction.) Because H is not Dirac Hermitian, (93) has the generic time-independent form in (24).

For modes that are energy eigenstates, (93) reduces to

$$(E_A - E_B) \int dx dy \hat{\psi}_B^L(x, y, t) \hat{\psi}_A^R(x, y, t) = 0, \quad (94)$$

so modes having unequal-energy eigenvalues are orthogonal as usual. When A is an eigenmode, the diagonal integrals $\int dx dy \hat{\psi}_A^L(x, y, t) \hat{\psi}_A^R(x, y, t)$ all vanish except for the ground-state eigenmode because all eigenstates other than the ground state have zero norm. The overlap integrals of the nonstationary wave functions, either with each other or with the stationary solutions, need not vanish because (93) requires that such overlaps be time independent, and not that they vanish. However, the equal-frequency wave functions are

constructed as limits of linear combinations of unequal-frequency modes, so the orthogonality of multiparticle eigenstates with different numbers of particles in the unequal-frequency theory translates into the orthogonality of the associated states in the equal-frequency case. Thus, even though not all of the equal-frequency theory states are stationary, both of the 1 and 1a states are orthogonal to all of the 2, 2a, 2b states, and so on.

Within any given multiparticle sector the overlaps need not all be zero. For the typical one-particle sector, for instance, we evaluate the overlap integrals directly and find that

$$\begin{aligned}\int dx dy \hat{\psi}_1^L(x, y, t) \hat{\psi}_1^R(x, y, t) &= 0, \\ \int dx dy \hat{\psi}_1^L(x, y, t) \hat{\psi}_{1a}^R(x, y, t) &= \int dx dy \hat{\psi}_{1a}^L(x, y, t) \hat{\psi}_1^R(x, y, t) = -\frac{\pi}{8\gamma^2\omega^4}, \\ \int dx dy \hat{\psi}_{1a}^L(x, y, t) \hat{\psi}_{1a}^R(x, y, t) &= -\frac{\pi}{8\gamma^2\omega^5}.\end{aligned}\tag{95}$$

This confirms that these overlaps are indeed time independent. The overlap integral $\int dx dy \hat{\psi}_1^L(x, y, t) \hat{\psi}_{1a}^R(x, y, t)$, for example, is time independent because the coefficient of the term that is linear in t is just the zero norm $\int dx dy \hat{\psi}_1^L(x, y, t) \hat{\psi}_1^R(x, y, t)$.

An alternative way to derive the relations in (95) is to take the $\epsilon \rightarrow 0$ limit of the normalization integrals N_1 and N_2 in (73). To order ϵ this yields

$$\begin{aligned}\int dx dy [\hat{\psi}_1^L(x, y) \hat{\psi}_1^R(x, y) - \epsilon \hat{\psi}_{1a}^L(x, y) \hat{\psi}_1^R(x, y) - \epsilon \hat{\psi}_1^L(x, y) \hat{\psi}_{1a}^R(x, y)] &= \frac{\pi\epsilon}{4\gamma^2\omega^4}, \\ \int dx dy [\hat{\psi}_1^L(x, y) \hat{\psi}_1^R(x, y) + \epsilon \hat{\psi}_{1a}^L(x, y) \hat{\psi}_1^R(x, y) + \epsilon \hat{\psi}_1^L(x, y) \hat{\psi}_{1a}^R(x, y)] &= -\frac{\pi\epsilon}{4\gamma^2\omega^4},\end{aligned}\tag{96}$$

from which the relevant relations in (95) follow. This procedure implies that in taking the $\epsilon \rightarrow 0$ limit of the time-independent normalization integrals of the unequal-frequency theory, we get time-independent expressions, some of which are nonzero, because as (73) and (76) show, the unequal-frequency-theory normalization integrals only vanish as powers of ϵ .

The nonstationary wave functions are not energy eigenstates, but at $t = 0$ the various multiparticle wave functions $\hat{\psi}_n^R(x, y, t)$ and $\hat{\psi}_{n,\alpha}^R(x, y, t)$ ($n = 0, 1, 2, \dots$, $\alpha = a, b, \dots$) contain polynomials in x and y of degree n , with just enough freedom to construct any arbitrary polynomial function of x and y . Hence any initial wave function $\hat{\psi}^R(x, y, t = 0)$ of x and y can be expanded in terms of a complete basis of polynomial wave functions: $\hat{\psi}^R(x, y, t = 0) = \sum_n a_n \hat{\psi}_n^R(x, y, t = 0) + \sum_{n,\alpha} a_{n,\alpha} \hat{\psi}_{n,\alpha}^R(x, y, t = 0)$.

The left wave functions have analogous expansions, with relations such as those in (95) implying that the quantity $\hat{N}(\hat{\psi}, t = 0) = \int dx dy \hat{\psi}^L(x, y, t = 0) \hat{\psi}^R(x, y, t = 0)$ is nonzero.

Thus, we can use the nonstationary solutions to construct any initial state whose initial probability $\hat{N}(\hat{\psi}, t = 0)$ is nonzero despite the presence of zero-norm energy eigenstates.

Each of the $\hat{\psi}_n^R(x, y, t = 0)$ and $\hat{\psi}_{n,\alpha}^R(x, y, t = 0)$ basis wave functions is a solution to the time-dependent Schrödinger equation, so each has a uniquely specified time evolution. Thus, given (93), we see that the probability integral $\int dx dy \hat{\psi}^L(x, y, t) \hat{\psi}^R(x, y, t)$ is preserved in time, with all the terms that involve powers of t dropping out of $\hat{N}(\hat{\psi}, t)$. Therefore, the equal-frequency theory is unitary.

To summarize the nature of completeness in the Jordan-block case, where there are fewer energy eigenstates than the dimensionality of the Hamiltonian, the missing eigenstates are replaced by an equal number of nonstationary solutions to the Schrödinger equation. Together the stationary and nonstationary states form a complete basis that may be used to construct an initial wave packet. Because these states are complete, the normalization of the wave packet is preserved in time.

Terms involving powers of t contribute to the Green's functions of the equal-frequency theory even though they play no role in probability integrals such as $\hat{N}(\hat{\psi}, t)$. We construct the equal-frequency Green's function $\hat{G}(x, y, x', y', t)$ as the limit of the unequal-frequency Green's function $G(x, y, x', y', t)$ given in (81). In the unequal-frequency theory the representative one-particle contribution to (81) is

$$\begin{aligned} & G(x, y, x', y', t)_{(1)} \\ &= \psi_1^R(x, y, t = 0) \frac{\mathcal{C}_1}{|N_1|} e^{-iE_1 t} \psi_1^L(x', y', t = 0) + \psi_2^R(x, y, t = 0) \frac{\mathcal{C}_2}{|N_2|} e^{-iE_2 t} \psi_2^L(x', y', t = 0) \\ &= \psi_0^R(x, y, t = 0) \psi_0^L(x', y', t = 0) e^{-i(E_0 + \omega)t} \frac{4\gamma^2 \omega^2 (\omega^2 - \epsilon^2)^{1/2}}{\pi \epsilon} \left[e^{-i\epsilon t} (\omega + \epsilon) \right. \\ &\quad \left. \times (x' - \omega y' + \epsilon y')(x + \omega y - \epsilon y) - e^{i\epsilon t} (\omega - \epsilon)(x' - \omega y' - \epsilon y')(x + \omega y + \epsilon y) \right]. \end{aligned} \quad (97)$$

Despite the presence of terms that behave as $1/\epsilon$, the $\epsilon \rightarrow 0$ limit of (97) exists:

$$\begin{aligned} G(x, y, x', y', t)_{(1)} &\rightarrow \hat{\psi}_0^R(x, y, t = 0) \hat{\psi}_0^L(x', y', t = 0) e^{-2i\omega t} \frac{8\gamma^2 \omega^3}{\pi} \\ &\quad \times [(1 - i\omega t)(x' - \omega y')(x + \omega y) + \omega y'(x + \omega y) - \omega(x' - \omega y')y], \end{aligned} \quad (98)$$

and using (85) and (88), we rewrite (98) as

$$\begin{aligned} \hat{G}(x, y, x', y', t)_{(1)} &= \frac{8\gamma^2 \omega^3}{\pi} \left[\hat{\psi}_1^R(x, y, t) \hat{\psi}_1^L(x', y', t = 0) - \omega \hat{\psi}_{1a}^R(x, y, t) \hat{\psi}_1^L(x', y', t = 0) \right. \\ &\quad \left. - \omega \hat{\psi}_1^R(x, y, t) \hat{\psi}_{1a}^L(x', y', t = 0) \right]. \end{aligned} \quad (99)$$

Equation (99) reveals the role played by the nonstationary states in the propagator. Because $\hat{G}(x, y, x', y', t)$ describes the propagation of a wave packet that is localized at (x, y) at an initial time t , both stationary and nonstationary wave functions are contained in the wave packet, and both are needed to form a complete basis with which to construct localized wave packets [30].

When the initial and final states are at the same time, the Green's function gives the normalization of the eigenstates of the position operator according to $\langle x, y, t | x', y', t \rangle = \delta(x - x')\delta(y - y')$. Consequently, the $\epsilon \rightarrow 0$ limit of the unequal-frequency completeness relation given in (78) must recover this property of (99). Explicit evaluation of (78) yields

$$\begin{aligned}
& \frac{8\gamma^2\omega^3}{\pi} \left[\hat{\psi}_1^R(x, y, t)\hat{\psi}_1^L(x', y', t) - \omega\hat{\psi}_{1a}^R(x, y, t)\hat{\psi}_1^L(x', y', t) - \omega\hat{\psi}_1^R(x, y, t)\hat{\psi}_{1a}^L(x', y', t) \right] \\
& + \frac{16\gamma^3\omega^4}{\pi} \left[\hat{\psi}_2^R(x, y, t)\hat{\psi}_2^L(x', y', t) - \omega\hat{\psi}_{2a}^R(x, y, t)\hat{\psi}_2^L(x', y', t) - \omega\hat{\psi}_2^R(x, y, t)\hat{\psi}_{2a}^L(x', y', t) \right] \\
& + \frac{8\gamma^3\omega^6}{\pi} \left[\hat{\psi}_{2a}^R(x, y, t)\hat{\psi}_{2a}^L(x', y', t) - \hat{\psi}_{2b}^R(x, y, t)\hat{\psi}_2^L(x', y', t) - \hat{\psi}_2^R(x, y, t)\hat{\psi}_{2b}^L(x', y', t) \right] + \dots \\
& = \delta(x - x')\delta(y - y')
\end{aligned} \tag{100}$$

at all times, just as required. Using (95) and the orthogonality of the different multiparticle wave functions, one can verify (100) by projecting with $\int dx dy \hat{\psi}_1^L(x, y, t)$, and so on. Equation (100) thus generalizes the standard completeness relation to the nonstationary case.

We have shown in this section how to develop a consistent quantum-mechanical theory given that the energy eigenstates do not form a complete basis. We did not make use of the fact that energy is quantized. Rather we introduced a specific canonical form for the commutators of the position and momentum operators, and we did this without reference to the structure of the Hamiltonian. Demanding that position and momentum operators such as x , p , y , and q be quantum operators required that we specify a Hilbert space on which they operate. In the basis in which the position operators are diagonal, we introduced a complete set of basis vectors $|x, y\rangle$, and in so doing we specified the Hilbert space once and for all.

An alternative but equivalent prescription is to represent the position and momentum operators as infinite-dimensional matrices acting on a Fock space, where all the states are created from a no-particle state. The raising operators then generate the complete set of basis vectors. This procedure works whether or not the Hamiltonian commutes with the

Fock-space number operator. This construction does not involve the Hamiltonian, and it does not depend on how many energy eigenstates the Hamiltonian possesses or on which states in the Fock space are its eigenvectors.

Normally, in quantum mechanics the Hamiltonian is a Hermitian operator whose energy eigenstates form a complete basis. This basis is in one-to-one correspondence with both the coordinate-space basis and the Fock-space basis. Hermitian Hamiltonians can be diagonalized, so for a Hermitian Hamiltonian it is advantageous to use its eigenvectors rather than any other set of vectors as the basis vectors. In this paper we have shown that one need not have a complete basis of energy eigenstates to characterize a quantum-mechanical Hilbert space when the Hamiltonian is in Jordan-block form. If the Hamiltonian cannot be diagonalized, the coordinate-space and Fock-space bases are central and the nonstationary solutions to the Schrödinger equation play a role that they do not play in the case of a Hermitian Hamiltonian. The lack of diagonalizability of Jordan-block Hamiltonians is not an impediment to the construction of a fully consistent and unitary Jordan-block quantum theory, and for such cases \mathcal{PT} quantum mechanics represents a distinct realization of quantum mechanics that exists in and of itself.

VIII. CONCLUSIONS AND COMMENTS

The fourth-order-derivative Pais-Uhlenbeck model is rich and instructive and we have used it to examine many issues of contemporary concern. In our previous paper [5] we showed that the unequal-frequency version of the model has a consistent quantum realization in which the spectrum is real and bounded below, the Hilbert space of states is ghost free, and time evolution is unitary. In this paper we have examined the case of the equal-frequency Pais-Uhlenbeck model. We have constructed this model by performing the equal-frequency limit of the unequal-frequency model. This limit is singular because the Hamiltonian develops a Jordan-block structure and many of the eigenstates of the Hamiltonian disappear. Nevertheless, we have shown that the limiting theory remains a consistent and unitary quantum theory.

Our solution to the equal-frequency Pais-Uhlenbeck model stems from our work on the unequal-frequency model. To find a physically acceptable realization of the unequal-frequency model we established in Ref. [5] that the Pais-Uhlenbeck model Hamiltonian H is

actually not Dirac Hermitian, but is instead \mathcal{PT} symmetric. This symmetry allowed us to construct a similarity transformation $\tilde{H} = e^{-\mathcal{Q}/2} H e^{\mathcal{Q}/2}$, which produces the Dirac Hermitian Hamiltonian \tilde{H} having the same eigenvalues as H . We calculated the operator \mathcal{Q} exactly and in closed form. The \mathcal{Q} operator reveals the singular nature of the equal-frequency limit because it ceases to exist in this limit. Thus, there is no equivalent Dirac-Hermitian Hamiltonian for the equal-frequency theory. Nevertheless, we have shown in this paper that all of the eigenfunctions that disappear in the equal-frequency limit are replaced by time-dependent solutions to the Schrödinger equation. As a result, completeness is maintained and the model continues to exhibit unitary time evolution. Thus, we have shown that the equal-frequency Pais-Uhlenbeck model is a unitary \mathcal{PT} quantum theory that has no equivalent Hermitian counterpart, and so \mathcal{PT} quantum mechanics should be regarded as being on an equal footing with standard Hermitian quantum mechanics, and as being completely independent of it at the special critical points where the operator \mathcal{Q} is singular.

Having shown that the Pais-Uhlenbeck Hamiltonian defines a physically acceptable quantum-mechanical theory, we believe that the techniques and results that we have described here will be of value in quantum field theory. The original motivation of Pais and Uhlenbeck was to see if one could avoid the renormalization infinities in theories such as quantum electrodynamics by having Feynman propagators that behave as $1/k^4$ rather than as $1/k^2$. The discouraging result that they found was that one could do so but at the price of having an energy spectrum without a lower bound. Subsequently, following the development of indefinite-metric theories, it was realized that one could evade this problem and have a spectrum that is bounded below, but one would have to pay a different and equally unpalatable price, namely, allowing states of negative Dirac norm and violating the physical requirement of unitarity. In our work on the Pais-Uhlenbeck model we have been able to overcome both the spectral nonpositivity and the norm nonpositivity (ghost) problems. Our optimism that ghost problems in quantum field theory can be solved is strengthened by the fact that \mathcal{PT} techniques have previously been used [11] to show that the quantum-field-theoretic Lee model is ghost free.

There are many possible directions for future research. Perhaps, the most intriguing future application of the ideas developed in this paper is to attempt to construct a consistent fourth-order derivative quantum theory of gravity in four spacetime dimensions. (Success in this endeavor might eliminate the need for ten-dimensional string theory.) Quantum field

theory is beyond the scope of this paper, but we note that the Pais-Uhlenbeck model serves as a quantum-mechanical prototype for higher-derivative theories, such as conformal gravity [31], which seek to construct a consistent quantum theory of gravity. Roughly speaking, the issues involved are illustrated by the fourth-order scalar field theory whose action is

$$I = \frac{1}{2} \int d^4x \left[\partial_\mu \partial_\nu \phi \partial^\mu \partial^\nu \phi - (M_1^2 + M_2^2) \partial_\mu \phi \partial^\mu \phi + M_1^2 M_2^2 \phi^2 \right]. \quad (101)$$

Here, the scalar field ϕ can be thought of as representing a typical component of the metric fluctuation $h_{\mu\nu}$ around a flat background $\eta_{\mu\nu}$. The action (101) gives the equation of motion

$$(\partial_t^2 - \nabla^2 + M_1^2)(\partial_t^2 - \nabla^2 + M_2^2)\phi(\bar{x}, t) = 0. \quad (102)$$

With $k^2 = k_0^2 - \bar{k}^2$, the propagator for this theory is given by

$$\begin{aligned} (\partial_t^2 - \nabla^2 + M_1^2)(\partial_t^2 - \nabla^2 + M_2^2)D^{(4)}(x - y) &= \delta^4(x - y), \\ D^{(4)}(k^2) &= \frac{1}{(k^2 - M_1^2)(k^2 - M_2^2)} = \frac{1}{M_1^2 - M_2^2} \left(\frac{1}{k^2 - M_1^2} - \frac{1}{k^2 - M_2^2} \right). \end{aligned} \quad (103)$$

This propagator exhibits the good $1/k^4$ convergence at large k^2 that is needed to make the quantum theory renormalizable, but it appears to do so at the expense of having ghost states. The identifications $\omega_1 = (\bar{k}^2 + M_1^2)^{1/2}$, $\omega_2 = (\bar{k}^2 + M_2^2)^{1/2}$ reduce (103) to (29), so there is hope that there might be a solution to the higher-derivative gravity ghost problem that parallels the solution to the Pais-Uhlenbeck-model ghost problem presented here and in Ref. [5].

To see how things could possibly develop, we note that for the generic higher-derivative scalar action $I = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi, \partial_\mu \partial_\nu \phi)$ the translation invariance of the theory leads to an energy-momentum tensor of the form

$$T_{\mu\nu} = \left[\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} - \partial_\lambda \left(\frac{\partial \mathcal{L}}{\partial \phi_{,\mu,\lambda}} \right) \right] \phi_{,\nu} + \frac{\partial \mathcal{L}}{\partial \phi_{,\mu,\lambda}} \phi_{,\nu,\lambda} - \eta_{\mu\nu} \mathcal{L}. \quad (104)$$

This energy-momentum tensor is conserved in field configurations that obey the fourth-order-derivative Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \right) + \partial_\mu \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial \phi_{,\mu,\nu}} \right) = 0. \quad (105)$$

In a phase space formulation of the theory, $T_{\mu\nu}$ is replaced by

$$T_{\mu\nu} = \pi_\mu \phi_{,\nu} + \pi_{\mu\lambda} \phi_{,\nu,\lambda} - \eta_{\mu\nu} \mathcal{L}. \quad (106)$$

Thus, for the particular action in (101) $T_{\mu\nu}$ has the form

$$T_{\mu\nu} = \pi_\mu \phi_{,\nu} + \pi_{\mu\lambda} \pi^\lambda_{\nu} - \frac{1}{2} \eta_{\mu\nu} \left[\pi_{\lambda\kappa} \pi^{\lambda\kappa} - (M_1^2 + M_2^2) \partial_\lambda \phi \partial^\lambda \phi + M_1^2 M_2^2 \phi^2 \right], \quad (107)$$

and the equation of motion is (102). For the metric $\eta_{\mu\nu} = (1, -1, -1, -1)$, the Hamiltonian of the theory is $H = \int d^3x T_{00}$, where

$$T_{00} = \pi_0 \dot{\phi} + \frac{1}{2} \pi_{00}^2 + \frac{1}{2} (M_1^2 + M_2^2) \dot{\phi}^2 - \frac{1}{2} M_1^2 M_2^2 \phi^2 - \frac{1}{2} \pi_{ij} \pi^{ij} + \frac{1}{2} (M_1^2 + M_2^2) \phi_{,i} \phi^{,i}. \quad (108)$$

We note that (108) is similar in structure to H_{PU} in (30), and (108) is its covariant generalization. Recalling the identifications $\omega_1 = (\bar{k}^2 + M_1^2)^{1/2}$, $\omega_2 = (\bar{k}^2 + M_2^2)^{1/2}$, the Hamiltonian (45) and commutation relations (46) covariantly generalize to

$$\begin{aligned} H &= \int d^3k \left[2(M_1^2 - M_2^2)(\bar{k}^2 + M_1^2) \hat{a}_{1,\bar{k}} a_{1,\bar{k}} + 2(M_1^2 - M_2^2)(\bar{k}^2 + M_2^2) \hat{a}_{2,\bar{k}} a_{2,\bar{k}} \right. \\ &\quad \left. + \frac{1}{2}(\bar{k}^2 + M_1^2)^{1/2} + \frac{1}{2}(\bar{k}^2 + M_2^2)^{1/2} \right], \\ [a_{1,\bar{k}}, \hat{a}_{1,\bar{k}'}] &= [2(M_1^2 - M_2^2)(\bar{k}^2 + M_1^2)^{1/2}]^{-1} \delta^3(\bar{k} - \bar{k}'), \\ [a_{2,\bar{k}}, \hat{a}_{2,\bar{k}'}] &= [2(M_1^2 - M_2^2)(\bar{k}^2 + M_2^2)^{1/2}]^{-1} \delta^3(\bar{k} - \bar{k}'), \\ [a_{1,\bar{k}}, a_{2,\bar{k}'}] &= 0, \quad [a_{1,\bar{k}}, \hat{a}_{2,\bar{k}'}] = 0, \quad [\hat{a}_{1,\bar{k}}, a_{2,\bar{k}'}] = 0, \quad [\hat{a}_{1,\bar{k}}, \hat{a}_{2,\bar{k}'}] = 0. \end{aligned} \quad (109)$$

In (109) all relative signs are positive.

Similarly, in the limiting case where $M_1^2 = M_2^2 = M^2$, the above equations are replaced by the covariant generalizations of the Jordan-block Hamiltonian (54) and the commutation relations (56):

$$\begin{aligned} H &= \int d^3k \left[8(\bar{k}^2 + M^2)^2 [2\hat{b}_{\bar{k}} b_{\bar{k}} + \hat{a}_{\bar{k}} b_{\bar{k}} + \hat{b}_{\bar{k}} a_{\bar{k}}] + (\bar{k}^2 + M^2)^{1/2} \right], \\ [a_{\bar{k}}, \hat{b}_{\bar{k}'}] &= [b_{\bar{k}}, \hat{a}_{\bar{k}'}] = [8(\bar{k}^2 + M^2)^{3/2}]^{-1} \delta^3(\bar{k} - \bar{k}'), \\ [a_{\bar{k}}, \hat{a}_{\bar{k}'}] &= 0, \quad [b_{\bar{k}}, \hat{b}_{\bar{k}'}] = 0, \quad [a_{\bar{k}}, b_{\bar{k}'}] = 0, \quad [\hat{a}_{\bar{k}}, \hat{b}_{\bar{k}'}] = 0. \end{aligned} \quad (110)$$

Now, there are zero-norm states and this continues to be the case even if we set $M^2 = 0$. Thus, we find zero-norm states in the pure fourth-order conformal gravity case where the action can schematically be represented by $I = \frac{1}{2} \int d^4x \partial_\mu \partial_\nu \phi \partial^\mu \partial^\nu \phi$ alone [32].

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APPENDIX A: LEHMANN SPECTRAL REPRESENTATION FOR HIGHER-DERIVATIVE THEORIES

To derive the Lehmann spectral representation in a higher-derivative field theory, we follow the same procedure as that for a second-order derivative field theory. The derivation makes use of three assumptions: the Poincaré transformation properties of the interacting fields, a completeness relation for the exact energy eigenstates of the interacting theory, and the existence of a stable ground state $|\Omega\rangle$ whose four-momentum we can take to be zero.

For the case of a self-interacting Hermitian scalar field $\phi(x)$, one introduces the translation generator $P_\mu = \int d^3x T_{0\mu}$, with the scalar field then transforming according to

$$[\phi(x), P_\mu] = i\partial_\mu\phi(x), \quad \phi(x) = e^{iP \cdot x} \phi(0) e^{-iP \cdot x}. \quad (\text{A1})$$

Given (A1), the matrix element of $\phi(x)$ between the vacuum and the one-particle state of four-momentum k_μ^n , positive k_0^n , and squared mass $m_n^2 = k^n \cdot k^n$ is:

$$\langle\Omega|\phi(x)|k_\mu^n\rangle = \langle\Omega|\phi(0)|k_\mu^n\rangle e^{-ik^n \cdot x}, \quad \langle k_\mu^n|\phi(x)|\Omega\rangle = \langle k_\mu^n|\phi(0)|\Omega\rangle e^{ik^n \cdot x}. \quad (\text{A2})$$

Let us provisionally take the completeness relation for the four-momentum eigenstates to be of the conventional Dirac form

$$\sum_n |n\rangle\langle n| = \mathbf{1}. \quad (\text{A3})$$

We can then write the two-point function of the field $\phi(x)$ in the form

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \sum_n |\langle\Omega|\phi(0)|k_\mu^n\rangle|^2 e^{-ik^n \cdot (x-y)}. \quad (\text{A4})$$

Introducing the spectral function $\rho(q^2)$ defined by

$$\rho(q^2) = (2\pi)^3 \sum_n \delta^4(k_\mu^n - q_\mu) |\langle\Omega|\phi(0)|k_\mu^n\rangle|^2 \theta(q_0), \quad (\text{A5})$$

we write the two-point function as

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty dm^2 \rho(m^2) \int \frac{d^4q}{(2\pi)^3} \theta(q_0) \delta(q^2 - m^2) e^{-iq \cdot (x-y)}. \quad (\text{A6})$$

Repeating the same analysis for the $\langle\Omega|\phi(y)\phi(x)|\Omega\rangle$ two-point function, we obtain the usual Lehmann representation

$$\begin{aligned} \Delta_{\text{F}}^{\text{int}}(x-y) &= i\langle\Omega|T[\phi(x)\phi(y)]|\Omega\rangle = i\langle\Omega|[\theta(x_0 - y_0)\phi(x)\phi(y) + \theta(y_0 - x_0)\phi(y)\phi(x)]|\Omega\rangle \\ &= \int_0^\infty dm^2 \rho(m^2) \Delta_{(\text{F},2)}^{\text{free}}(x-y; m^2), \end{aligned} \quad (\text{A7})$$

for the time-ordered product of the interacting fields [19]. Here, $\Delta_{(F,2)}^{\text{free}}(x-y; m^2) = [i\langle\Omega|T[\phi(x)\phi(y)]|\Omega\rangle]_{\text{free}}$ is the Feynman propagator for a free scalar field with mass m , namely the propagator that obeys the free second-order-theory relations

$$[\partial_t^2 - \nabla^2 + m^2] \Delta_{(F,2)}^{\text{free}}(x-y; m^2) = \delta^4(x-y), \quad \Delta_{(F,2)}^{\text{free}}(k^2; m^2) = -\frac{1}{k^2 - m^2}. \quad (\text{A8})$$

(Comparing (A8) with the fourth-order propagator in (103), we note a relative minus sign in Fourier space.)

Equation (A7) expresses the exact Feynman propagator of the interacting theory as a spectral integral over free second-order Feynman propagators with a continuum of mass values. No assumption has been made regarding the order of the equation of motion obeyed by the interacting field $\phi(x)$. Nevertheless, the free propagator $\Delta_{(F,2)}^{\text{free}}(x-y; m^2)$ in (A7) still satisfies a second-order differential equation. Independent of the structure or order of the interacting field equation, the mass-shell condition associated with the eigenstates of the exact four-momentum operator P_μ is still a second-order condition because of Poincaré invariance.

The derivation of (A7) is generic, and we can apply it to a field theory having a fourth-order field equation. To do so we need to show that in the absence of interactions the quantity $i\langle\Omega|T[\phi(x)\phi(y)]|\Omega\rangle$ can indeed be identified with the free fourth-order propagator

$$D^{(4)}(k^2) = \frac{1}{M_1^2 - M_2^2} \left(\frac{1}{k^2 - M_1^2} - \frac{1}{k^2 - M_2^2} \right) \quad (\text{A9})$$

introduced above. In the free fourth-order case associated with the equation of motion

$$(\partial_t^2 - \nabla^2 + M_1^2)(\partial_t^2 - \nabla^2 + M_2^2)\phi(\bar{x}, t) = 0, \quad (\text{A10})$$

we must evaluate the relevant time derivatives of the two-field T-product.

For the first time derivative we obtain

$$\begin{aligned} \frac{\partial}{\partial x_0} [\langle\Omega|T[\phi(x)\phi(y)]|\Omega\rangle] &= \langle\Omega| [\theta(x_0 - y_0)\dot{\phi}(x)\phi(y) + \theta(y_0 - x_0)\phi(y)\dot{\phi}(x)] |\Omega\rangle \\ &+ \langle\Omega|\delta(x_0 - y_0)[\phi(x), \phi(y)]|\Omega\rangle, \end{aligned} \quad (\text{A11})$$

and, with $\delta(x_0 - y_0)[\phi(x), \phi(y)] = 0$, we obtain

$$\frac{\partial}{\partial x_0} [\langle\Omega|T[\phi(x)\phi(y)]|\Omega\rangle] = \langle\Omega| [\theta(x_0 - y_0)\dot{\phi}(x)\phi(y) + \theta(y_0 - x_0)\phi(y)\dot{\phi}(x)] |\Omega\rangle. \quad (\text{A12})$$

Analogously, for the second time derivative we obtain

$$\begin{aligned} \frac{\partial^2}{\partial x_0^2} [\langle \Omega | T[\phi(x)\phi(y)] | \Omega \rangle] &= \langle \Omega | \left[\theta(x_0 - y_0) \ddot{\phi}(x) \phi(y) + \theta(y_0 - x_0) \phi(y) \ddot{\phi}(x) \right] | \Omega \rangle \\ &+ \langle \Omega | \delta(x_0 - y_0) [\dot{\phi}(x), \phi(y)] | \Omega \rangle. \end{aligned} \quad (\text{A13})$$

However, unlike the second-order case, $\phi(x)$ and $\dot{\phi}(x)$ are not canonical conjugates. Rather, they are the covariant analogs of the y and $\dot{y} = i[H, y] = -ix$ variables of the fourth-order Pais-Uhlenbeck oscillator theory. With the y and x operators being commuting variables, the fourth-order equal-time commutator $\delta(x_0 - y_0) [\dot{\phi}(x), \phi(y)]$ must thus vanish. Consequently, (A13) reduces to

$$\frac{\partial^2}{\partial x_0^2} [\langle \Omega | T[\phi(x)\phi(y)] | \Omega \rangle] = \langle \Omega | \left[\theta(x_0 - y_0) \ddot{\phi}(x) \phi(y) + \theta(y_0 - x_0) \phi(y) \ddot{\phi}(x) \right] | \Omega \rangle. \quad (\text{A14})$$

For the third time derivative we obtain

$$\begin{aligned} \frac{\partial^3}{\partial x_0^3} [\langle \Omega | T[\phi(x)\phi(y)] | \Omega \rangle] &= \langle \Omega | \left[\theta(x_0 - y_0) \frac{\partial^3 \phi(x)}{\partial x_0^3} \phi(y) + \theta(y_0 - x_0) \phi(y) \frac{\partial^3 \phi(x)}{\partial x_0^3} \right] | \Omega \rangle \\ &+ \langle \Omega | \delta(x_0 - y_0) [\ddot{\phi}(x), \phi(y)] | \Omega \rangle. \end{aligned} \quad (\text{A15})$$

Because $\phi(x)$ and $\ddot{\phi}(x)$ are the covariant analogs of the y and $-i\dot{x} = [H, x] = -ip/\gamma$ variables, the commutator $\delta(x_0 - y_0) [\ddot{\phi}(x), \phi(y)]$ also vanishes. Thus, (A15) reduces to

$$\frac{\partial^3}{\partial x_0^3} [\langle \Omega | T[\phi(x)\phi(y)] | \Omega \rangle] = \langle \Omega | \left[\theta(x_0 - y_0) \frac{\partial^3 \phi(x)}{\partial x_0^3} \phi(y) + \theta(y_0 - x_0) \phi(y) \frac{\partial^3 \phi(x)}{\partial x_0^3} \right] | \Omega \rangle. \quad (\text{A16})$$

Finally, for the fourth time derivative we obtain

$$\begin{aligned} \frac{\partial^4}{\partial x_0^4} [\langle \Omega | T[\phi(x)\phi(y)] | \Omega \rangle] &= \langle \Omega | \left[\theta(x_0 - y_0) \frac{\partial^4 \phi(x)}{\partial x_0^4} \phi(y) + \theta(y_0 - x_0) \phi(y) \frac{\partial^4 \phi(x)}{\partial x_0^4} \right] | \Omega \rangle \\ &+ \langle \Omega | \delta(x_0 - y_0) \left[\frac{\partial^3 \phi(x)}{\partial x_0^3}, \phi(y) \right] | \Omega \rangle. \end{aligned} \quad (\text{A17})$$

However, $\partial^3 \phi(x) / \partial x_0^3$ is the analog of $-ip/\gamma = [H, p]/\gamma = i(\omega_1^2 + \omega_2^2)x + q/\gamma$, and the commutator of y and q is $[y, q] = i$. Thus, the equal-time commutator $\delta(x_0 - y_0) [\partial^3 \phi(x) / \partial x_0^3, \phi(y)]$ is equal to $-i\delta^4(x - y)$. Hence, the i times the T-product obeys (103), and $i\langle \Omega | T[\phi(x)\phi(y)] | \Omega \rangle$ is the fourth-order-theory Green's function we need.

Note that in the fourth-order case in Fourier space the left side of (A7) behaves as $1/k^4$ at large k^2 . However, the Fourier transform of $\Delta_{(\text{F},2)}^{\text{free}}(x - y; m^2)$ behaves as $1/k^2$. The spectral

function $\rho(m^2)$ in (A5) is positive definite. Since (A7) is a mathematical identity, we thus have a contradiction. The difference between the large k^2 behaviors of $\Delta_{(F,4)}^{\text{free}}(x-y)$ and of $\Delta_{(F,2)}^{\text{free}}(x-y)$ implies that the spectral function $\rho(m^2)$ is not positive definite. Hence, the standard Dirac completeness relation in (A3) for the energy eigenstates cannot be valid in the fourth-order case. This analysis immediately generalizes to all higher-order derivative theories for which the propagator is even more convergent at large k^2 . We conclude that before constructing the Hilbert space appropriate to a higher-derivative theory, we know from the outset that the needed inner product cannot be the standard Dirac one.

A simple modification of the Dirac norm that gives a nonpositive definite $\rho(m^2)$ would be to replace (A3) by

$$\sum_n \eta_n |n\rangle \langle n| = \mathbf{1}, \quad (\text{A18})$$

where η_n is ± 1 , and to replace (A5) by

$$\rho(q^2) = (2\pi)^3 \sum_n \delta^4(k_\mu^n - q_\mu) |\langle \Omega | \phi(0) | k_\mu^n \rangle|^2 \eta_n \theta(q_0). \quad (\text{A19})$$

However, this choice would violate unitarity.

When the Hamiltonian is not Hermitian, positivity of the spectral weight function is no longer mandatory [33]. However, we need not forego unitarity because we no longer use the Dirac inner product. Instead, we use the \mathcal{PT} inner product

$$\langle n | e^{-\mathcal{Q}} | m \rangle = \delta_{m,n}, \quad \sum_n |n\rangle \langle n| e^{-\mathcal{Q}} = \mathbf{1} \quad (\text{A20})$$

introduced earlier. With such an inner product the spectral function of (A5) is replaced by

$$\rho(q^2) = (2\pi)^3 \sum_n \delta^4(k_\mu^n - q_\mu) \langle \Omega | e^{-\mathcal{Q}} \phi(0) | k_\mu^n \rangle \langle k_\mu^n | e^{-\mathcal{Q}} \phi(0) | \Omega \rangle \theta(q_0). \quad (\text{A21})$$

Unlike the spectral function associated with a Dirac norm, this spectral function need not be positive definite. To evaluate it, in analogy to (44) we set

$$\phi(x) = \int \frac{d^3 k}{(2\pi)^{3/2}} \left[-i a_{1,\bar{k}} e^{-ik^1 \cdot x} + a_{2,\bar{k}} e^{-ik^2 \cdot x} - i \hat{a}_{1,\bar{k}} e^{ik^1 \cdot x} + \hat{a}_{2,\bar{k}} e^{ik^2 \cdot x} \right]. \quad (\text{A22})$$

We define right and left vacua according to

$$a_{1,\bar{k}} |0_R\rangle = 0, \quad a_{2,\bar{k}} |0_R\rangle = 0, \quad \langle 0_L | \hat{a}_{1,\bar{k}} = 0, \quad \langle 0_L | \hat{a}_{2,\bar{k}} = 0, \quad (\text{A23})$$

and in terms of the notation in (A23) identify

$$\begin{aligned} |\Omega\rangle &= |0_R\rangle, & |k_\mu^i\rangle &= [2(M_1^2 - M_2^2)(\bar{k}^2 + M_i^2)^{1/2}]^{1/2} \hat{a}_{i,\bar{k}} |0_R\rangle, \\ \langle \Omega | e^{-\mathcal{Q}} &= \langle 0_L |, & \langle k_\mu^i | &= [2(M_1^2 - M_2^2)(\bar{k}^2 + M_i^2)^{1/2}]^{1/2} \langle 0_L | a_{i,\bar{k}}, \end{aligned} \quad (\text{A24})$$

as normalized according to

$$\langle 0_L | 0_R \rangle = 1, \quad \langle k_\mu^1 | k_\mu^1 \rangle = 1, \quad \langle k_\mu^2 | k_\mu^2 \rangle = 1. \quad (\text{A25})$$

Through use of the commutation relations (109) we obtain the on-shell contributions

$$\begin{aligned} \langle \Omega | e^{-\mathcal{Q}} \phi(0) | k_\mu^1 \rangle \langle k_\mu^1 | e^{-\mathcal{Q}} \phi(0) | \Omega \rangle &= -\frac{1}{(2\pi)^3 2(\bar{k}^2 + M_1^2)^{1/2}(M_1^2 - M_2^2)}, \\ \langle \Omega | e^{-\mathcal{Q}} \phi(0) | k_\mu^2 \rangle \langle k_\mu^2 | e^{-\mathcal{Q}} \phi(0) | \Omega \rangle &= \frac{1}{(2\pi)^3 2(\bar{k}^2 + M_2^2)^{1/2}(M_1^2 - M_2^2)}, \end{aligned} \quad (\text{A26})$$

to give us precisely the relative minus sign we require. Recalling the minus sign in (A8), insertion of (A26) in the Lehmann representation then yields the propagator (A9). Hence, in fourth-order theories compatibility of the Lehmann representation with unitarity is readily achievable [34].

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- [1] C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998); C. M. Bender, S. Boettcher, and P. N. Meisinger, J. Math. Phys. **40**, 2201 (1999).
 - [2] C. M. Bender, D. C. Brody, and H. F. Jones, Phys. Rev. Lett. **89**, 270401 (2002); Am. J. Phys. **71**, 1095 (2003); Phys. Rev. Lett. **93**, 251601 (2004).
 - [3] C. M. Bender, Contemp. Phys. **46**, 277 (2005) and Rep. Prog. Phys. **70**, 947 (2007).
 - [4] A. Pais and G. E. Uhlenbeck, Phys. Rev. **79**, 145 (1950).
 - [5] C. M. Bender and P. D. Mannheim, Phys. Rev. Lett. **100**, 110402 (2008).
 - [6] P. Dorey, C. Dunning and R. Tateo, J. Phys. A **34**, L391 (2001) and **34**, 5679 (2001); Czech. J. Phys. **54**, 35 (2004); arXiv: hep-th/0201108.
 - [7] P. Dorey, C. Dunning, and R. Tateo, J. Phys. A: Math. Gen. **40**, R205 (2007).
 - [8] F. G. Scholtz, H. B. Geyer, and F. J. W. Hahne, Ann. Phys. **213**, 74 (1992).
 - [9] A. Mostafazadeh, J. Math. Phys. **43**, 205 (2002) and **43**, 3944 (2002); J. Phys. A: Math. Gen. **36**, 7081 (2003).
 - [10] T. D. Lee, Phys. Rev. **95**, 1329 (1954).
 - [11] C. M. Bender, S. F. Brandt, J.-H. Chen, and Q. Wang, Phys. Rev. D **71**, 025014 (2005); H. F. Jones, arXiv: hep-th/0711.4967.
 - [12] M. S. Swanson, J. Math. Phys. **45**, 585 (2004).
 - [13] C. M. Bender and H. F. Jones, arXiv:0709.3605v1 [hep-th].

- [14] C. M. Bender, H. F. Jones, and R. J. Rivers, Phys. Lett. **B625**, 333 (2005).
- [15] A. A. Andrianov, Ann. Phys. (N.Y.) **140**, 82 (1982); V. Buslaev and V. Grecchi, J. Phys. A: Math. Gen. **26**, 5541 (1993); H. F. Jones and J. Mateo, Phys. Rev. D **73**, 085002 (2006); C. M. Bender, D. C. Brody, J.-H. Chen, H. F. Jones, K. A. Milton and M. C. Ogilvie, Phys. Rev. D **74**, 025016 (2006); H. F. Jones, J. Mateo, and R. J. Rivers, Phys. Rev. D **74**, 125022 (2006).
- [16] C. M. Bender and D. W. Hook, arXiv:0802.2910v1 [hep-th].
- [17] D. N. Christodoulides, private communication.
- [18] Jordan showed that via a sequence of similarity transformations any N -dimensional square matrix can be brought to a form where it is either diagonal or in triangular Jordan-block form in which all of the elements on one side of the diagonal are zero. Since the nonzero elements on the other side of the diagonal do not contribute to the secular equation, the N elements on the diagonal of a Jordan-block form matrix are then the N eigenvalues of the matrix. When all of the eigenvalues are real, a Jordan-block matrix is an example of a non-Hermitian matrix having eigenvalues that are all real. (Hermiticity is only a sufficient condition for obtaining real eigenvalues, not a necessary one. The eigenvalues of a Hermitian matrix are real, but there is nothing that requires the eigenvalues of a non-Hermitian matrix to be complex. Non-Hermitian matrices may have real eigenvalues.) Because Jordan-block matrices cannot be brought to a diagonal form by any further similarity transform, they possess fewer than N eigenvectors and the eigenvectors of a Jordan block matrix do not form a complete basis.
- [19] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).
- [20] P. D. Mannheim and A. Davidson, arXiv: hep-th/0001115.
- [21] P. D. Mannheim and A. Davidson, Phys. Rev. A **71**, 042110 (2005).
- [22] P. D. Mannheim, Found. Phys. **37**, 532 (2007).
- [23] A path-integral representation of $G^{\text{UNC}}(t)$ cannot exist because the complex-frequency contour chosen to evaluate $G^{\text{UNC}}(t)$ would not regulate the path integral. Specifically, if we replace ω_1^2 and ω_2^2 by the generic $\omega_1^2 - i\alpha_1\epsilon$ and $\omega_2^2 - i\alpha_2\epsilon$ in (34), we would generate a real term in the path-integral action of the form $(\gamma/2) \int dt [-(\alpha_1 + \alpha_2)\epsilon \dot{z}^2 + (\omega_1^2\alpha_2 + \omega_2^2\alpha_1)\epsilon z^2]$, with a successful regulation requiring that this term be negative definite on every path. Since the path integral must be integrated over independent z and \dot{z} paths, the unconventional choice $\alpha_1 = 1$, $\alpha_2 = -1$ would leave the \dot{z} path integration unregulated, but the conventional Feynman choice

$\alpha_1 = 1, \alpha_2 = 1$ does regulate the \hat{z} path integration. Hence, we only consider $G^{\text{F}}(t)$. However, it also is not yet fully regulated because its z path integration has no convergence factor. We resolve this issue by identifying the *Stokes wedges* that cause the z^2 term in the action to be replaced by a term that is pure imaginary. The regular Feynman contour prescription then regulates the entire path integral.

- [24] A physically equivalent alternative to making the similarity transformation in (37) is to keep the original operator basis of (30) and to take the quantum-mechanical operators \hat{z} and \hat{p}_z to be anti-Hermitian on the real- z axis. The $[\hat{z}, \hat{p}_z] = i$ commutator would then be realized via $\hat{z} = -iz, \hat{p}_z = \partial_z$ with the real coordinate z . In this case the ground-state eigenfunction would be given by $\psi_0(z, x) = \exp [-(\gamma/2)(\omega_1 + \omega_2)\omega_1\omega_2 z^2 + \gamma\omega_1\omega_2 zx - (\gamma/2)(\omega_1 + \omega_2)x^2]$ and it would be well behaved on the real- z and real- x axes. Alternatively, for the unbounded-below energy spectrum associated with the unconventional propagator $G^{\text{UNC}}(t)$ of (33) the state of energy $E'_0 = (\omega_1 - \omega_2)/2$ solves the Schrödinger equation (35) with eigenfunction $\psi'_0(z, x) = \exp [-(\gamma/2)(\omega_1 - \omega_2)\omega_1\omega_2 z^2 - i\gamma\omega_1\omega_2 zx - (\gamma/2)(\omega_1 - \omega_2)x^2]$; this eigenfunction is well behaved on the real- z and real- x axes. The unbounded-below energy sector can thus be associated with the quantum-mechanical operators \hat{z} and \hat{p}_z , which are Hermitian on the real- z axis, while the bounded-below energy sector can be associated with the quantum-mechanical operators \hat{z} and \hat{p}_z , which are anti-Hermitian on the real- z axis. The two candidate realizations of the Pais-Uhlenbeck model correspond to different contour prescriptions in the complex-energy plane, and they are also distinguished by the differing domains of Hermiticity that they assign to the position and momentum operators. To underscore this point, we recall that for the generic one-dimensional harmonic oscillator $\hat{H} = \hat{v}^2 + \hat{u}^2$, with $[\hat{u}, \hat{v}] = i$, the energy spectrum is manifestly bounded below if the operators $\hat{u} = (a + a^\dagger)/\sqrt{2}$ and $\hat{v} = i(a^\dagger - a)/\sqrt{2}$ are Hermitian on the real- u axis; the state $|\Omega\rangle$ that a annihilates has energy $E_0 = 1$ and the normalizable ground-state eigenfunction $\psi_0(u) = e^{-u^2/2}$. However, for this oscillator the Schrödinger equation also admits a solution associated with the state $|\Omega'\rangle$ that a^\dagger annihilates, and it has negative energy $\tilde{E}_0 = -1$ and eigenfunction $\tilde{\psi}_0(u) = e^{u^2/2}$. This eigenfunction is not normalizable on the real- u axis, but is normalizable on the imaginary- u axis. Moreover, in this sector the excited state $a|\Omega'\rangle$ is an energy eigenstate with a Dirac norm $\langle\Omega'|a^\dagger a|\Omega'\rangle$, which is negative. The eigenfunctions in this sector would become normalizable on the real axis and their norms would become positive if the operators \hat{u} and \hat{v} were anti-Hermitian on the real

axis. Thus obtaining non-normalizable wave functions or negative Dirac norms serves as a warning that certain operators are not Hermitian on the real axis. The choice of domain of Hermiticity of the position and momentum operators determines which of the two realizations (bounded or unbounded energy spectrum) the harmonic oscillator takes. While all norms are positive in both of these domains when they are constructed properly, in only one (Hermitian on the real u -axis) is the energy spectrum bounded below. An analogous situation occurs for the two-oscillator Pais-Uhlenbeck model except that this time the operators of one of the two oscillators would need to be anti-Hermitian on the real axis to obtain an energy spectrum that is both bounded-below and free of negative norm states.

- [25] The correlator $\langle \Omega | e^{-\mathcal{Q}} y(t) y(0) | \Omega \rangle$ obeys the fourth-order derivative Pais-Uhlenbeck wave equation (27), but after the $e^{-\mathcal{Q}/2}$ transformation the correlator $\langle \tilde{\Omega} | \tilde{y}(t) \tilde{y}(0) | \tilde{\Omega} \rangle$ will obey the fourth-order equation too even though $\langle \tilde{\Omega} | y(t) y(0) | \tilde{\Omega} \rangle$ obeys a second order wave equation. While H and \tilde{H} have the same energy eigenspectra, the correlators of interest to the Pais-Uhlenbeck theory are those associated with H rather than those associated with \tilde{H} .
- [26] As well as deriving (45) and (46) starting from (42) and (43), we can also derive these relations from the form for \tilde{H} given in (41). In terms of $\tilde{A}_{\pm} = (2\gamma\omega_1)^{-1/2}(p \pm i\gamma\omega_1 x)$ and $\tilde{B}_{\pm} = (2\gamma\omega_1^2\omega_2)^{-1/2}(q \pm i\gamma\omega_1^2\omega_2 y)$, which obey $[\tilde{A}_-, \tilde{A}_+] = [\tilde{B}_-, \tilde{B}_+] = 1$, $[\tilde{A}_{\pm}, \tilde{B}_{\pm}] = [\tilde{A}_{\pm}, \tilde{B}_{\mp}] = 0$, \tilde{H} diagonalizes as $\tilde{H} = \omega_1 \tilde{A}_+ \tilde{A}_- + \omega_2 \tilde{B}_+ \tilde{B}_- + (\omega_1 + \omega_2)/2$. On transforming according to $A_{\pm} = e^{\mathcal{Q}/2} \tilde{A}_{\pm} e^{-\mathcal{Q}/2} = [2\gamma\omega_1(\omega_1^2 - \omega_2^2)]^{-1/2}(\omega_1 p + i\gamma\omega_1\omega_2^2 y \pm i\gamma\omega_1^2 x \pm q)$, $B_{\pm} = e^{\mathcal{Q}/2} \tilde{B}_{\pm} e^{-\mathcal{Q}/2} = [2\gamma\omega_2(\omega_1^2 - \omega_2^2)]^{-1/2}(q + i\gamma\omega_2^2 x \pm i\gamma\omega_1^2\omega_2 y \pm \omega_2 p)$ and defining $A_- = -a_1[2\gamma\omega_1(\omega_1^2 - \omega_2^2)]^{1/2}$, $A_+ = -\hat{a}_1[2\gamma\omega_1(\omega_1^2 - \omega_2^2)]^{1/2}$, $B_- = -ia_2[2\gamma\omega_2(\omega_1^2 - \omega_2^2)]^{1/2}$, $B_+ = i\hat{a}_2[2\gamma\omega_2(\omega_1^2 - \omega_2^2)]^{1/2}$, (45) and (46) then follow. We thus confirm the pattern of coefficients in (44) and the absence of negative energies in (45) or ghost-sign commutators in (46).
- [27] Some earlier studies of \mathcal{PT} -symmetric Hamiltonians of Jordan-block form may be found in A. Mostafazadeh, J. Math. Phys. **43**, 6343 (2002) [Erratum: J. Math. Phys. **44**, 943 (2003)]; G. Sclarici and L. Solombrino, J. Math. Phys. **44**, 4450 (2003); A. Blasi, G. Sclarici, and L. Solombrino, J. Phys. A: Math. Gen. **37**, 4335 (2004); U. Günther and F. Stefani, Czech. J. Phys. **55**, 1099 (2005); A. V. Sokolov, A. A. Andrianov, and F. Cannata, J. Phys. A: Math. Gen. **39**, 10207 (2006). A case where a Jordan-block form is associated with a non-Dirac Hermitian Hamiltonian whose eigenvalues are real may be found in the spin quantum Hall effect studies of J. B. Marston and S.-W. Tsai, Phys. Rev. Lett. **82**, 4906 (1999) and

- T. Senthil, J. B. Marston, and M. P. A. Fisher, Phys. Rev. B **60**, 4245 (1999). We are indebted to Dr. S.-W. Tsai for informing us about these latter two papers.
- [28] Appropriately modified, the discussion given here parallels that given in [22].
- [29] The presence of nonstationary wave functions in Jordan-Block theories has been discussed in A. P. Seyranian and A. A. Mailybaev, *Multiparameter stability theory with mechanical applications*, Series on Stability, Vibration and Control of Systems, Vol. 13, Series A (World Scientific, River Edge, NJ, 2003); H. Baumgärtel, *Analytic perturbation theory for matrices and operators*, Operator Theory: Advances and Applications, Vol. 15 (Birkhäuser, Stuttgart, 1985); M. V. Keldysh, Russ. Math. Surv. **26**, 15 (1971). We thank Dr. U. Günther for alerting us to these references.
- [30] The need for the nonstationary $\hat{\psi}_{1a}(x, y, t)$ for localization purposes was also noted in [22], where the deep-Euclidean-time limit of the equal-frequency path integral (the equal-frequency limit of (34)) was found to yield an asymptotic term of the form $\tau e^{-2\omega\tau}$.
- [31] P. D. Mannheim, Prog. Part. Nucl. Phys. **56**, 340 (2006).
- [32] We have obtained a consistent physical interpretation of the Pais-Uhlenbeck oscillator model by treating some of the position and momentum operators as being anti-Hermitian rather than Hermitian on the real axis. At first glance, such an interpretation might not seem acceptable for the gravitational field because one would think that the gravitational field should be real. This is not necessarily so because in replacing $g_{\mu\nu}$ by $ig_{\mu\nu}$ (and thus $g^{\mu\nu}$ by $-ig^{\mu\nu}$ to maintain $g^{\mu\sigma}g_{\sigma\nu} = \delta^\mu_\nu$), neither the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ nor the components of the $R^\lambda_{\mu\nu\sigma}$ Riemann tensor would be affected. In four space-time dimensions $\det(g_{\mu\nu})$ would not be affected either. Even though contractions of $R^\lambda_{\mu\nu\sigma}$ would generate factors of i , all such factors could be absorbed by redefining the overall multiplicative coefficients in the gravitational action. Thus, the pure gravitational sector would then not be affected at all. Introducing such factors of i might lead to observable consequences as a result of an interaction between an anti-Hermitian gravitational field and some standard Hermitian field, such as the electromagnetic field, and this could lead to possibly observable interference effects. For a discussion of the interplay between Hermitian and non-Hermitian Hamiltonians see [13], and for a discussion of the propagation of a light wave through an interferometer which is rotated in the earth's gravitational field (the optical analog of the Colella-Overhauser-Werner effect) see P. D. Mannheim, Phys. Rev. A **57**, 1260 (1998) and Found. Phys. **26**, 1683 (1996). It could be that the solution

to quantum gravity, if it is described by a fourth-order theory such as conformal gravity, is that the gravitational field is anti-Hermitian rather than Hermitian. If the pure fourth-order Hamiltonian possesses only zero-norm and nonstationary states, there might be no observable on-shell graviton.

- [33] C. M. Bender, S. Boettcher, P. N. Meisinger, and Q. Wang, *Phys. Lett. A* **302**, 286 (2002).
- [34] We are indebted to Dr. A. M. Polyakov for suggesting that we look at the unitarity problem for fourth-order theories from the perspective of the Lehmann representation.